

# A More Powerful Two-Sample Test in High Dimensions using Random Projection

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## Abstract

We study the hypothesis testing problem of detecting a shift between the means of two multivariate normal distributions in the high-dimensional setting, allowing for the data dimension  $p$  to exceed the sample size  $n$ . Specifically, we propose a new test statistic for the two-sample test of means that integrates a random projection with the classical Hotelling  $T^2$  statistic. Working under a high-dimensional framework with  $(p, n) \rightarrow \infty$ , we first derive an asymptotic power function for our test, and then provide sufficient conditions for it to achieve greater power than other state-of-the-art tests. Lastly, using ROC curves generated from simulated data, we demonstrate superior performance with competing tests in the parameter regimes anticipated by our theoretical results.

## 1 Introduction

The goal of two-sample hypothesis testing is to determine whether or not two data sets  $\{X_1, \dots, X_{n_1}\}$  and  $\{Y_1, \dots, Y_{n_2}\}$  have been sampled from the same distribution. In the high-dimensional version of this problem, each sample is a vector in  $\mathbb{R}^p$ , and the dimension  $p$  may be substantially larger than both of the sample sizes  $n_1$  and  $n_2$ . This situation makes the two-sample problem challenging, since the cumulative effect of variance in many variables can “explain away” the correct hypothesis. Moreover, the occurrence of high-dimensional two-sample problems is becoming increasingly common in application domains such as molecular biology and fMRI [e.g., 1, 2, 3, 4]. In transcriptomics, for instance,  $p$  gene expression measures on the order of hundreds or thousands may be used to investigate differences between two biological conditions, and it is often difficult to obtain sample sizes  $n_1$  and  $n_2$  larger than several dozen in each condition. For problems such as these, classical methods may be ineffective, or not applicable at all. Accordingly, there has been growing interest in developing testing procedures that are better suited to deal with the effects of dimension [e.g., 5, 6, 7, 8, 9].

A fundamental instance of the general two-sample problem is the two-sample test of means with Gaussian data. In this case, two independent sets of samples  $\{X_1, \dots, X_{n_1}\}$  and  $\{Y_1, \dots, Y_{n_2}\} \subset \mathbb{R}^p$  are generated in an i.i.d. manner from  $p$ -dimensional multivariate normal distributions  $N(\mu_1, \Sigma)$  and  $N(\mu_2, \Sigma)$  respectively, where the mean vectors  $\mu_1$  and  $\mu_2$ , and positive-definite covariance matrix  $\Sigma \succ 0$ , are all fixed and unknown. The hypothesis testing problem of interest is

$$\mathbf{H}_0 : \mu_1 = \mu_2 \text{ versus } \mathbf{H}_1 : \mu_1 \neq \mu_2. \quad (1)$$

The most well-known test statistic for this problem is the Hotelling  $T^2$  statistic, defined by

$$T^2 := \frac{n_1 n_2}{n_1 + n_2} (\bar{X} - \bar{Y})^\top \hat{\Sigma}^{-1} (\bar{X} - \bar{Y}), \quad (2)$$

where  $\bar{X} := \frac{1}{n_1} \sum_{j=1}^{n_1} X_j$  and  $\bar{Y} := \frac{1}{n_2} \sum_{j=1}^{n_2} Y_j$  are the sample means,  $\hat{\Sigma}$  is the pooled sample covariance matrix, given by  $\hat{\Sigma} := \frac{1}{n} \sum_{j=1}^{n_1} (X_j - \bar{X})(X_j - \bar{X})^\top + \frac{1}{n} \sum_{j=1}^{n_2} (Y_j - \bar{Y})(Y_j - \bar{Y})^\top$ , and we define  $n := n_1 + n_2 - 2$  for convenience.

When  $p > n$ , the matrix  $\hat{\Sigma}$  is singular, and the Hotelling test is not well-defined. Even when  $p \leq n$ , the Hotelling test is known to perform poorly if  $p$  is nearly as large as  $n$ . This behavior was demonstrated in a seminal paper of Bai and Saranadasa [5] (or BS for short), who studied the performance of the Hotelling test under  $(p, n) \rightarrow \infty$  with  $p/n \rightarrow 1 - \epsilon$ , and showed that the asymptotic power of the test suffers for small values of  $\epsilon > 0$ . In subsequent years, a number of improvements on the Hotelling test in the high-dimensional setting have been proposed [e.g., 5, 6, 7, 8].

Due to the well-known degradation of  $\hat{\Sigma}$  as an estimate of  $\Sigma$  in high dimensions, the line of research on extensions of the Hotelling test has focused on replacing  $\hat{\Sigma}$  in the definition of  $T^2$  with other estimators of  $\Sigma$ . For instance, BS [5] proposed a test statistic based on the quantity  $(\bar{X} - \bar{Y})^\top (\bar{X} - \bar{Y})$ , which can be viewed as replacing  $\hat{\Sigma}$  with  $I_{p \times p}$ . It was shown by BS that their statistic achieves non-trivial asymptotic power whenever the ratio  $p/n$  converges to a constant  $c \in (0, \infty)$ . This statistic was later refined by Chen and Qin [8] (CQ for short) who showed that the same asymptotic power can be achieved without imposing any explicit restriction on the limit of  $p/n$ . Another direction was considered by Srivastava and Du [6, 7] (SD for short), who proposed a test statistic based on  $(\bar{X} - \bar{Y})^\top \hat{D}^{-1} (\bar{X} - \bar{Y})$ , where  $\hat{D}$  is the diagonal matrix associated with  $\hat{\Sigma}$ , i.e.  $\hat{D}_{ii} = \hat{\Sigma}_{ii}$ . This choice ensures that  $\hat{D}$  is invertible for all dimensions  $p$  with probability 1. Srivastava and Du demonstrated that their test for problem (1) has superior asymptotic power to the tests of BS and CQ under a particular parameter setting and local alternative when  $n = \mathcal{O}(p)$ . To the best of our knowledge, the procedures of CQ and SD represent the state-of-the-art among tests for problem (1)<sup>1</sup> with a known asymptotic power function under the scaling  $(p, n) \rightarrow \infty$ .

In this paper, we propose a new testing procedure for problem (1) in the high-dimensional setting, which involves randomly projecting the  $p$ -dimensional samples into a space of lower dimension  $k \leq \min\{n, p\}$ , computing the Hotelling test statistic in  $\mathbb{R}^k$ , and then averaging over the ensemble of projection matrices. Working within a high-dimensional framework that allows  $p/n$  to tend to a positive constant or infinity, we derive the asymptotic power function for our test, and show that under certain conditions, it can outperform the tests of BS, CQ, and SD in terms of asymptotic relative efficiency.

From a conceptual point of view, the procedure studied here differs from past work in the way that covariance structure is incorporated into the test statistic. The previously described test statistics of BS, CQ, and SD are essentially based on versions of the Hotelling  $T^2$  test using diagonal estimators of  $\Sigma$ . Our analysis and simulations show that this limited estimation of  $\Sigma$  sacrifices power when the data variables are correlated, or when most of the variance can be captured in a small number of variables. In this regard, our procedure is motivated by the idea that covariance structure may be used more effectively by averaging over a projected version of the pooled sample covariance matrix. We note that the use of projection-based approaches to two-sample testing and covariance estimation have also been considered previously by Jacob et al. [10], as well as Cl  men  on et al. [9], Cuesta-Albertos et al. [11], and Marzetta et al. [12].

The remainder of this paper is organized as follows. In Section 2, we discuss the intuition for

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<sup>1</sup>The tests of BS, CQ, and SD actually extend somewhat beyond (1) in that their asymptotic power functions have been obtained under data-generating distributions more general than Gaussian, e.g. satisfying simple moment conditions. The analysis of CQ also allows for the data-generating distributions to have unequal covariance matrices.

our testing procedure, and then formally define the test statistic. Section 3 is devoted to a number of theoretical results about the performance of the test. Theorems 1 and 2 in Section 3 characterize the asymptotic distribution of our test statistic under the null and alternative hypotheses, and formulas for critical values and asymptotic power are provided. Theorems 3 and 4 in Sections 3.4 and 3.5 give sufficient conditions for achieving greater power than the tests of CQ and SD in the sense of asymptotic relative efficiency. In Section 4, we use simulated data to make ROC curve comparisons with the mentioned tests, as well as some recent non-parametric procedures such as maximum mean discrepancy (MMD) [13], kernel Fisher discriminant analysis (KFDA) [14], and a test based on area-under-curve maximization, denoted TreeRank [9]. These simulations show that our test outperforms competing tests in the parameter regimes anticipated by our theoretical results.

**Notation.** Let  $\delta := \mu_1 - \mu_2$  denote the *shift vector* between the distributions  $N(\mu_1, \Sigma)$  and  $N(\mu_2, \Sigma)$ , and define the ordered pair of parameters  $\theta := (\delta, \Sigma)$ . For a positive-definite covariance matrix  $\Sigma$ , let  $D_\sigma$  be the diagonal matrix obtained by setting the off-diagonal entries of  $\Sigma$  to 0, and define the associated correlation matrix  $R := D_\sigma^{-1/2} \Sigma D_\sigma^{-1/2}$ . Let  $z_{1-\alpha}$  denote the  $1 - \alpha$  quantile of the standard normal distribution, and let  $\Phi$  be its cumulative distribution function. If  $A$  is a matrix in  $\mathbb{R}^{p \times p}$ , let  $\|A\|_{\text{op}}$  denote its operator norm (maximum singular value), and define the Frobenius norm  $\|A\|_F := \sqrt{\sum_{i,j} A_{ij}^2}$ . When all the eigenvalues of  $A$  are real, we denote them by  $\lambda_{\min}(A) = \lambda_p(A) \leq \dots \leq \lambda_1(A) = \lambda_{\max}(A)$ . We use the notation  $W_m$  to denote an  $m \times m$  white Wishart matrix, i.e.  $W_m \sim W_m(n, I_{m \times m})$ . The notation  $f(n) \lesssim g(n)$  or  $f(n) = \mathcal{O}(g(n))$  means that there is some absolute constant  $c \in (0, \infty)$  such that the inequality  $f(n) \leq c g(n)$  holds for all large  $n$ . If both  $f(n) \lesssim g(n)$  and  $g(n) \lesssim f(n)$  hold, then we write  $f(n) \asymp g(n)$ . The notation  $f(n) = o(g(n))$  means  $f(n)/g(n) \rightarrow 0$  as  $n \rightarrow \infty$ . The symbols  $\stackrel{d}{=}$  and  $\stackrel{d}{\rightarrow}$  refer to equality and convergence in distribution.

## 2 Random projection method

For the remainder of the paper, we retain the setup for the two-sample test of means (1) with Gaussian data given in Section 1. In particular, our procedure can be implemented with  $p > n$  or  $p \leq n$ , as long as  $k$  is chosen such that  $k \leq \min\{n, p\}$ , with  $n := n_1 + n_2 - 2$ . This bound on  $k$  will be assumed without comment, as well as the condition  $k \rightarrow \infty$  as  $(n, p) \rightarrow \infty$ . In Section 3.3, we demonstrate an optimality property of the choice  $k = \lfloor n/2 \rfloor$ , which is valid in moderate or high-dimensions, i.e.,  $p \geq \lfloor n/2 \rfloor$ , and we restrict our attention to this case in our comparison results (Theorems 3 and 4).

### 2.1 Intuition for random projection method

To provide some intuition for our method, it is possible to consider the problem (1) in terms of a competition between the dimension  $p$ , and the “statistical distance” separating  $\mathbf{H}_0$  and  $\mathbf{H}_1$ . On one hand, the accumulation of variance from a large number of variables makes it difficult to discriminate between the hypotheses, and thus, it is desirable to reduce the dimension of the data. On the other hand, methods for reducing dimension also tend to bring  $\mathbf{H}_0$  and  $\mathbf{H}_1$  “closer together”, making them harder to distinguish. Mindful of the fact that the Hotelling  $T^2$  measures

the separation of  $\mathbf{H}_0$  and  $\mathbf{H}_1$  in terms of the Kullback-Leibler (KL) divergence [15, p. 216, eq. 12]  $D_{\text{KL}}(N(\mu_1, \Sigma) \| N(\mu_2, \Sigma)) = \frac{1}{2} \delta^\top \Sigma^{-1} \delta$ , with  $\delta = \mu_1 - \mu_2$ , we see that the relevant statistical distance is driven by the length of  $\delta$ . Consequently, we seek to transform the data in a way that reduces dimension and preserves most of the length of  $\delta$  upon passing to the transformed distributions. From this geometric point of view, it is natural to exploit the fact that random projections can simultaneously reduce dimension and approximately preserve length with high probability [16].

If a projection matrix  $P_k^\top \in \mathbb{R}^{k \times p}$  is used to project data from  $\mathbb{R}^p$  to  $\mathbb{R}^k$ , and the classical Hotelling  $T^2$  statistic is constructed from the projected data, then the resulting statistic is proportional to  $(\bar{X} - \bar{Y})^\top [P_k(P_k^\top \hat{\Sigma} P_k)^{-1} P_k^\top] (\bar{X} - \bar{Y})$ . In particular, if  $k \leq \min\{n, p\}$  and  $P_k$  is drawn from the Haar distribution on the Steifel manifold  $\mathbb{V}_{p,k} := \{P_k \in \mathbb{R}^{p \times k} : P_k^\top P_k = I_{k \times k}\}$ , then the matrix  $P_k^\top \hat{\Sigma} P_k$  is invertible with probability 1, which ensures that this is a well-defined statistic. Hence, our consideration of the distance-preserving property of random projections naturally leads to using the matrix  $P_k(P_k^\top \hat{\Sigma} P_k)^{-1} P_k^\top$  as a surrogate for  $\hat{\Sigma}^{-1}$  in the high-dimensional setting. Moreover, to eliminate the variability of a single random projection, this idea can be refined by computing the average of the matrix  $P_k(P_k^\top \hat{\Sigma} P_k)^{-1} P_k^\top$  over the ensemble  $P_k$ , to any desired degree of precision. The resulting statistic is proportional to  $(\bar{X} - \bar{Y})^\top \mathbb{E}_{P_k} [P_k(P_k^\top \hat{\Sigma} P_k)^{-1} P_k^\top] (\bar{X} - \bar{Y})$ , and this quantity will be the subject of the remainder of the paper.<sup>2</sup> The key advantage of this approach is that the matrix  $\mathbb{E}_{P_k} [P_k(P_k^\top \hat{\Sigma} P_k)^{-1} P_k^\top]$ , retains some information about the off-diagonal entries of  $\Sigma$ , whereas a diagonal estimate makes no essential use of correlation structure. Indeed, this intuition is confirmed by our theoretical results, which quantify the relationship between the degree of correlation and the power that is gained over statistics based on diagonal estimates of  $\Sigma$ .

## 2.2 Implementation of testing procedure

For an integer  $k \in \{1, \dots, \min\{n, p\}\}$ , let  $\mathcal{V}_{p,k}$  denote the Haar distribution on the set of matrices  $\{P_k \in \mathbb{R}^{p \times k} : P_k^\top P_k = I_{k \times k}\}$ . If  $P_k$  is drawn from  $\mathcal{V}_{p,k}$ , independently of the data, then our random projection-based test statistic is defined by

$$\bar{T}_k^2 := \frac{n_1 n_2}{n_1 + n_2} (\bar{X} - \bar{Y})^\top \mathbb{E}_{P_k} [P_k(P_k^\top \hat{\Sigma} P_k)^{-1} P_k^\top] (\bar{X} - \bar{Y}). \quad (3)$$

For a desired nominal level  $\alpha \in (0, 1)$ , our testing procedure rejects the null hypothesis  $\mathbf{H}_0$  if and only if  $\bar{T}_k^2 \geq t_\alpha$ , where

$$t_\alpha := \frac{y_n}{1 - y_n} n + \sqrt{\frac{2y_n}{(1 - y_n)^3}} \sqrt{n} z_{1-\alpha},$$

$y_n := k/n$ , and  $z_{1-\alpha}$  is the  $1 - \alpha$  quantile of the standard normal distribution. The fact that  $t_\alpha$  is an asymptotically level- $\alpha$  critical value follows from Theorem 1 of Section 3, where it is shown that  $\bar{T}_k^2$  is asymptotically normal under  $\mathbf{H}_0$ .

In order implement our testing procedure, the user must choose a value for  $k$ , as well as a method for computing  $\mathbb{E}_{P_k} [P_k(P_k^\top \hat{\Sigma} P_k)^{-1} P_k^\top]$ . Our analysis in Section 3.3 shows that the choice  $k = \lfloor n/2 \rfloor$  possesses an optimality property under certain conditions, and our simulation results confirm that this is a reasonable default choice. Nevertheless, p-values can be obtained with any choice of  $k \in \{1, \dots, \min\{n, p\}\}$  using our asymptotic theory. Concerning the computation of the matrix  $\mathbb{E}_{P_k} [P_k(P_k^\top \hat{\Sigma} P_k)^{-1} P_k^\top]$ , it is a basic fact that if  $G \in \mathbb{R}^{p \times k}$  is a random matrix with i.i.d.  $N(0, 1)$  entries, and if  $G = QR$  is a *thin QR decomposition* [17, p.230], then  $Q$  is distributed according

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<sup>2</sup>The paper [12] also considers the matrix  $\mathbb{E}_{P_k} [P_k(P_k^\top \hat{\Sigma} P_k)^{-1} P_k^\top]$ , from a different point of view.

to  $\mathcal{V}_{p,k}$ , and satisfies the algebraic identity  $G(G^\top \widehat{\Sigma} G)^{-1} G^\top = Q(Q^\top \widehat{\Sigma} Q)^{-1} Q^\top$  (see Lemma 4 in Section B.1 for additional details). Consequently,  $\bar{T}_k^2$  may be computed as an average over matrices with i.i.d.  $N(0, 1)$  entries, which are computationally inexpensive to generate. However, our analysis will make use of the distribution  $\mathcal{V}_{p,k}$ .

A second issue is the the number of independent copies of  $P_k$  to use when averaging, and this choice will depend on the data set at hand, as well as the degree of precision desired in the calculation of p-values. To eliminate the effects of finite averaging in the calculation of p-values, it is possible to examine the fluctuation of p-values when using a fixed number of projections  $N$ , and then increase  $N$  until the fluctuations become negligible. The stabilization of p-values with increasing values of  $N$  is illustrated in Figure 5 of Section 4. Furthermore, our simulations in Figures 3 and 4 of Section 4 show that in many cases, the ROC curve of the  $\bar{T}_k^2$  statistic stabilizes after averaging over only 30 projections.

### 3 Main results and their consequences

Our first two main results in Theorems 1 and 2 characterize the critical values and asymptotic power function for our statistic  $\bar{T}_k^2$ , and they are proved in Appendices B and C. Theorems 3 and 4 to follow provide comparisons of asymptotic relative efficiency with tests proposed in past work, and they are proved in Sections 3.4 and 3.5.

#### 3.1 Critical values and asymptotic power function

As is standard in high-dimensional asymptotics, we consider a sequence of hypothesis testing problems indexed by  $n$ , allowing the dimension  $p$ , sample sizes  $n_1$  and  $n_2$ , mean vectors  $\mu_1$  and  $\mu_2$  and covariance matrix  $\Sigma$  to implicitly vary as functions of  $n$ . Although we assume that  $y_n := k/n$  tends to a limit in the interval  $(0, 1)$ , the derivation of our critical values and asymptotic power function does not restrict  $p/n$  to tend to a finite limit.

**(A1)** There is a constant  $y \in (0, 1)$  such that  $y_n = y + o(\frac{1}{\sqrt{n}})$ .

To state our result on the asymptotic normality of  $\bar{T}_k^2$  under the null hypothesis, we define the numerical sequences  $\bar{\mu}_n := \frac{y_n}{1-y_n}n$  and  $\bar{\sigma}_n := \sqrt{\frac{2y_n}{(1-y_n)^3}}\sqrt{n}$ .

**Theorem 1** (Critical values). *Assume that the null hypothesis  $\mathbf{H}_0$  and condition (A1) hold. Then as  $(n, p) \rightarrow \infty$  we have the the limit*

$$\frac{\bar{T}_k^2 - \bar{\mu}_n}{\bar{\sigma}_n} \xrightarrow{d} N(0, 1), \quad (4)$$

and consequently, the critical value  $t_\alpha$  satisfies

$$\mathbb{P}(\bar{T}_k^2 \geq t_\alpha) = \alpha + o(1).$$

To consider the alternative hypothesis  $\mathbf{H}_1$  and obtain an asymptotic power function, we will assume that the ratio  $n_1/n$  tends to a limit in condition (A2), and that a local alternative holds in condition (A3). This is the same as the local alternative used by Bai and Saranadasa [5] to study the asymptotic power of the classical Hotelling test under  $(n, p) \rightarrow \infty$ . The condition (A3)

carries the meaning that the KL divergence between the  $p$ -dimensional sampling distributions,  $D_{\text{KL}}(N(\mu_1, \Sigma) \parallel N(\mu_2, \Sigma)) = \frac{1}{2} \delta^\top \Sigma^{-1} \delta$ , tends to 0. Similar local alternatives are used elsewhere in [5, 6, 7, 8] with consideration to the high-dimensional problem (1).

(A2) There is a constant  $b \in (0, 1)$  such that  $\frac{n_1}{n} = b + o(\frac{1}{\sqrt{n}})$ .

(A3) (Local alternative.) The shift vector and covariance matrix satisfy  $\delta^\top \Sigma^{-1} \delta = o(1)$ .

To set some notation for our asymptotic power result in Theorem 2, let  $\theta := (\delta, \Sigma)$  denote an ordered pair containing the relevant parameters for problem (1), and define  $\beta(\theta)$  as the exact (non-asymptotic) power function of the test  $\bar{T}_k^2$  at level  $\alpha$ . We also define  $\Delta_k$  as twice the Kullback-Leibler divergence between the projected sampling distributions,

$$\Delta_k := 2 D_{\text{KL}} \left( N(P_k^\top \mu_1, P_k^\top \Sigma P_k) \parallel N(P_k^\top \mu_2, P_k^\top \Sigma P_k) \right) = \delta^\top P_k (P_k^\top \Sigma P_k)^{-1} P_k^\top \delta. \quad (5)$$

Note that this quantity is a random variable, due to its dependence on the random matrix  $P_k^\top$ . By taking an average over  $P_k$ , we obtain the deterministic quantity

$$\bar{\Delta}_k := \mathbb{E}_{P_k} [\Delta_k],$$

which determines the asymptotic power of  $\bar{T}_k^2$ , according to the following theorem.

**Theorem 2** (Asymptotic power function). *Assume conditions (A1), (A2), and (A3). Then, as  $(n, p) \rightarrow \infty$ , the power function  $\beta(\theta)$  satisfies*

$$\beta(\theta) = \Phi \left( -z_{1-\alpha} + b(1-b) \sqrt{\frac{1-y}{2y}} \bar{\Delta}_k \sqrt{n} \right) + o(1). \quad (6)$$

**Remarks.** Notice that if  $\bar{\Delta}_k = 0$  (e.g. under  $\mathbf{H}_0$ ), then  $\Phi(-z_{1-\alpha} + 0) = \alpha$ , which corresponds to blind guessing at level  $\alpha$ . Consequently, the second term  $b(1-b) \sqrt{\frac{1-y}{2y}} \bar{\Delta}_k \sqrt{n}$  determines the advantage of our procedure over blind guessing. Since  $\bar{\Delta}_k$  is twice the KL divergence between the projected sampling distributions, these observations conform to the intuition from Section 2 that the KL divergence measures the discrepancy between  $\mathbf{H}_0$  and  $\mathbf{H}_1$ .

### 3.2 Asymptotic relative efficiency (ARE)

Having obtained an asymptotic power function in Theorem 2, we are now in position to provide a detailed comparison with the tests of Chen and Qin [8] and Srivastava and Du [6, 7]. We denote the asymptotic power function of our level- $\alpha$  random projection-based test (RP) by

$$\beta_{\text{RP},k}(\theta) := \Phi \left( -z_{1-\alpha} + b(1-b) \sqrt{\frac{1-y}{2y}} \bar{\Delta}_k \sqrt{n} \right). \quad (7)$$

The asymptotic power functions for the level- $\alpha$  testing procedures of CQ [8] and SD [6, 7] are given by

$$\beta_{\text{CQ}}(\theta) := \Phi \left( -z_{1-\alpha} + \frac{b(1-b)}{\sqrt{2}} \frac{\|\delta\|_2^2 n}{\|\Sigma\|_F} \right), \quad \text{and} \quad (8a)$$

$$\beta_{\text{SD}}(\theta) := \Phi \left( -z_{1-\alpha} + \frac{b(1-b)}{\sqrt{2}} \frac{\delta^\top D_\sigma^{-1} \delta n}{\|R\|_F} \right), \quad (8b)$$

where  $D_\sigma$  denotes the matrix formed by setting the off-diagonal entries of  $\Sigma$  to 0, and  $R$  denotes the correlation matrix associated to  $\Sigma$ . The functions  $\beta_{\text{CQ}}$  and  $\beta_{\text{SD}}$  are derived under local alternatives and asymptotic assumptions that are similar to the ones used here to obtain  $\beta_{\text{RP},k}$ . In particular, all three functions can be obtained allowing  $p/n$  to tend to an arbitrary positive constant, or to infinity.

A standard method of comparing asymptotic power functions is through the concept of *asymptotic relative efficiency*, or ARE for short (e.g., see van der Vaart [18, ch. 14-15]). Since the term added to  $-z_{1-\alpha}$  inside the  $\Phi$  function is what controls power, the relative efficiency of tests is defined by the ratio of such terms. More explicitly, we define

$$\text{ARE}(\beta_{\text{CQ}}/\beta_{\text{RP},k}) := \left( \frac{\|\delta\|_2^2 n}{\|\Sigma\|_F} \sqrt{\frac{1-y}{y}} \bar{\Delta}_k \sqrt{n} \right)^2, \quad \text{and} \quad (9a)$$

$$\text{ARE}(\beta_{\text{SD}}/\beta_{\text{RP},k}) := \left( \frac{\delta^\top D_\sigma^{-1} \delta n}{\|R\|_F} \sqrt{\frac{1-y}{y}} \bar{\Delta}_k \sqrt{n} \right)^2. \quad (9b)$$

With these definitions, whenever the ARE is less than 1, the method proposed in this paper has greater asymptotic power than the competing test—with the advantage being greater for smaller values of the ARE. As discussed at more detail later, Theorems 3 and 4 provide conditions on the shift  $\delta$  and covariance matrix  $\Sigma$  under which the ARE is small.

For arbitrary values of the parameters  $\Sigma$  and  $\delta$ , the ARE formulas (9) are not easy to interpret. In order to gain insight into the conditions that determine the relative performance of the RP, CQ, and SD tests, we will consider two natural cases where the ARE can be reduced to an interpretable form. Since  $\text{ARE}(\beta_{\text{CQ}}/\beta_{\text{RP},k})$  and  $\text{ARE}(\beta_{\text{SD}}/\beta_{\text{RP},k})$  are invariant with respect to scaling of  $\delta$ , the orientation  $\delta/\|\delta\|_2$  is the only part of the shift vector that is relevant for comparing power. Likewise, it is natural to consider the extreme cases of a “uniform” orientation and an “adversarial” orientation. To encode the idea of a *uniform shift*, for which no direction is of particular importance, we will allow  $\delta/\|\delta\|_2$  to follow the uniform (Haar) distribution on the unit sphere of  $\mathbb{R}^p$ . A similar assumption was considered by Srivastava and Du [6], who let  $\delta$  be a deterministic vector with all coordinates equal to the same value, in order to compare with the results of BS [5]. To encode the idea of an adversarial shift, we will consider a situation in which  $\delta$  is likely to point in a direction of high variance, i.e. a direction in which the shift is easily explained away by chance variation. One way this can be carried out is to allow  $\delta$  to follow the distribution  $N(0, s_n \Sigma)$  for some scaling factor  $s_n$ , since the most likely direction is parallel to the eigenvector corresponding to the largest eigenvalue of  $\Sigma$ , and this will be referred to as a *tilted shift*. We emphasize that our testing procedure does not rely on these choices of the distribution of  $\delta$ , and that our asymptotic power function in Theorem 2 is derived under the assumption of fixed and unknown values of  $\delta$  and  $\Sigma$ .

In classical analyses of asymptotic relative efficiency, the ARE is usually a deterministic quantity that does not depend on  $n$ . However, in the current context, our use of high-dimensional asymptotics, as well as a randomly chosen value of  $\delta$ , lead to an ARE that can be regarded as a sequence of random variables indexed by  $n$ . To be clear about the meaning of Propositions 1 and 2, and Theorems 3 and 4 below, we point out that  $\delta$  is the only source of randomness in the ARE. We complete the preparation for our comparison theorems by stating Propositions 1 and 2, which describe the scaling of the power-determining quantity  $\bar{\Delta}_k$  under a uniform shift and a tilted shift. These are the main technical tools for our comparison results, and proofs can be found in Appendix A. Although these scaling results do require some assumptions on the eigenvalues of  $\Sigma$ , these assumptions essentially just amount to requiring that the maximum and minimum eigenval-

ues do not converge to 0 or  $\infty$  too quickly. In particular, we do not require the eigenvalues to be bounded away from 0 or  $\infty$ .

**Proposition 1** (Scaling of  $\overline{\Delta}_k$  with a uniform shift).

Assume that the shift  $\delta$  has a spherical distribution  $\mathbb{P}_\delta$  with  $\mathbb{P}_\delta(\delta = 0) = 0$ . Then, the following limit statements hold with  $(n, p) \rightarrow \infty$ .

(i) If  $\frac{1}{\lambda_{\min}(\Sigma)k} \frac{\text{tr}(\Sigma)}{p} = o(1)$ , then for any  $c \in (0, 1)$ , we have

$$\mathbb{P}_\delta \left( \frac{\overline{\Delta}_k}{\|\delta\|_2^2} \geq \frac{ck}{\text{tr}(\Sigma)} \right) \rightarrow 1. \quad (10)$$

(ii) If  $\Sigma = I_{p \times p} + \Gamma$  for some matrix  $\Gamma \succeq 0$  with  $\text{rank}(\Gamma) = o(k)$ , then

$$\frac{\overline{\Delta}_k}{\|\delta\|_2^2} \bigg/ \frac{k}{p} \rightarrow 1 \quad \text{in probability under } \mathbb{P}_\delta. \quad (11)$$

**Remarks.** The case  $\Sigma = I_{p \times p} + \Gamma$  includes a variety of sparsity patterns of the matrix  $\Sigma$ . For instance, if we choose  $v$  to be a vector with support set  $S \subset \{1, \dots, p\}$ , then using  $\Gamma = vv^\top$  gives a matrix  $\Sigma$  whose off-diagonal entries are supported on  $S \times S$ . By choosing  $\Gamma$  to be a sum of rank 1 matrices of this type, more complex block-correlation patterns can be captured.

**Proposition 2** (Scaling of  $\overline{\Delta}_k$  with a tilted shift).

Assume that the shift  $\delta$  has a distribution  $\mathbb{P}_\delta = N(0, s_n \Sigma)$  for some positive scaling factor  $s_n$ , and that  $\|\Sigma\|_{op} / \text{tr}(\Sigma) = o(1)$ . Then, as  $(n, p) \rightarrow \infty$ , we have

$$\frac{\overline{\Delta}_k}{\|\delta\|_2^2} \bigg/ \frac{k}{\text{tr}(\Sigma)} \rightarrow 1 \quad \text{in probability under } \mathbb{P}_\delta. \quad (12)$$

### 3.3 Choice of projection dimension $k = \lfloor n/2 \rfloor$

We now demonstrate an optimality property of the choice of projected dimension  $k = \lfloor n/2 \rfloor$ . Letting  $k/n \rightarrow y \in (0, 1)$ , recall that the asymptotic power function from Theorem 2 is

$$\beta_{\text{RP},k}(\theta) = \Phi \left( -z_{1-\alpha} + b(1-b) \sqrt{\frac{1-y}{2y}} \overline{\Delta}_k \sqrt{n} \right).$$

Since Propositions 1 and 2 indicate that  $\overline{\Delta}_k$  scales linearly in  $k$  up to random fluctuations, we see that formally replacing  $k$  with  $yn$  leads to maximizing the function  $f(y) := \sqrt{\frac{1-y}{2y}} y$ . The fact that  $f$  is maximized at  $y = 1/2$  suggests that  $k = \lfloor n/2 \rfloor$  may be optimal in certain cases. The following proposition formalizes this idea by providing conditions under which this choice is optimal in the sense of ARE. For this purpose, we define  $k^* := \lfloor n/2 \rfloor$ ,  $y^* := 1/2$ , and

$$\text{ARE}(\beta_{\text{RP},k} / \beta_{\text{RP},k^*}) := \frac{\frac{1-y}{2y} \overline{\Delta}_k^2}{\frac{1-y^*}{2y^*} \overline{\Delta}_{k^*}^2}.$$



Note that Propositions 1 and 2 provide two classes of examples where the conditions of the following result are satisfied (along subsequences).

**Proposition 3.** *Assume that either of the limits  $\frac{\bar{\Delta}_k}{\|\delta\|_2^2} / \frac{k}{\text{tr}(\Sigma)} \rightarrow 1$  or  $\frac{\bar{\Delta}_k}{\|\delta\|_2^2} / \frac{k}{p} \rightarrow 1$ , hold for some sequence of parameters  $\delta$  and  $\Sigma$ . Then, for any choice of  $k$  with  $k/n \rightarrow y \in (0, 1)$ , we have the optimality property*

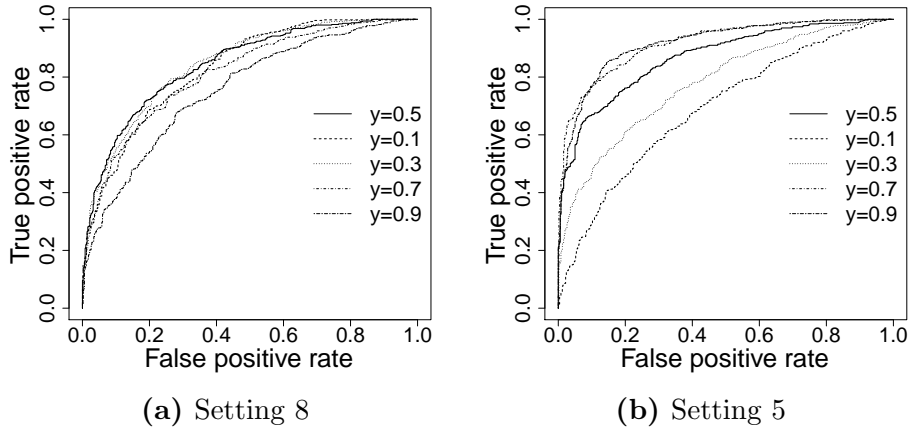
$$\lim_{(n,p) \rightarrow \infty} \text{ARE}(\beta_{\text{RP},k} / \beta_{\text{RP},k^*}) \leq 1.$$

*Proof.* Under either of the limits, it is straightforward to check that for all  $y \in (0, 1)$ , the numerical sequence  $\text{ARE}(\beta_{\text{RP},k} / \beta_{\text{RP},k^*})$  tends to  $4y(1-y)$ , which is at most 1.  $\square$

**Remarks.** The ROC curves in Figure 1 illustrate several choices of projection dimension, with  $k = \lfloor yn \rfloor$  and  $y = 0.1, 0.3, 0.5, 0.7, 0.9$ , under two different parameter settings. We refer to these as Setting 5 and 8, as they are described in Section 4.1, where full details are provided. In all cases, the statistic  $\bar{T}_k^2$  is computed by averaging over 100 projections.

In Setting 8, the matrix  $\Sigma$  is constructed with a slowly decaying spectrum, and a matrix of eigenvectors drawn from the uniform (Haar) distribution on the orthogonal group (to induce a non-diagonal  $\Sigma$ ). Also, the shift vector  $\delta$  is drawn from  $N(0, s_n \Sigma)$  with  $s_n = \sqrt{\text{tr}(\Sigma)}/2$ , in accordance with the conditions of Proposition 2. As expected from Proposition 2, we see that the ROC curve corresponding to  $y = 1/2$  dominates the others up to small fluctuations.

In Setting 5, the matrix  $\Sigma$  is constructed to have a block-diagonal form, and the shift vector  $\delta$  is drawn from the uniform distribution on the unit sphere, which results in a set of conditions similar to those of Proposition 1[(ii)]. The ROC curves in Figure 1 show that  $y = 1/2$  is not far from optimal among the five choices of  $y$ .



**Figure 1.** Setting 5 corresponds to a randomly generated covariance matrix with a slowly decaying spectrum, and a shift vector drawn according to  $N(0, s \Sigma)$  with  $s = \sqrt{\text{tr}(\Sigma)}/2$ . Setting 8 corresponds to a block-diagonal covariance matrix with a shift vector drawn from the uniform distribution on the unit sphere. We see that the choice  $k = \lfloor n/2 \rfloor$ , corresponding to  $y = 1/2$ , is nearly optimal in both cases. Full details can be found in Section 4.1.

### 3.4 Power comparison with CQ

We are now equipped to compare the asymptotic power of our test with the test of Chen and Qin (CQ) [8] in the following theorem. A proof is given at the end of this section (3.4).

**Theorem 3.** *Assume the conditions of either Proposition 1[(i)] (uniform shift) or Proposition 2 (tilted shift), and choose  $k = \lfloor n/2 \rfloor$ . Fix a number  $\epsilon_1 > 0$ , and let  $c_1(\epsilon_1)$  be any constant strictly greater than  $4/\epsilon_1$ . Then, as long as the inequality*

$$n \geq c_1(\epsilon_1) \frac{\text{tr}(\Sigma)^2}{\|\Sigma\|_F^2}, \quad (13)$$

*holds for all large  $n$ , we have the limit  $\mathbb{P}_\delta[\text{ARE}(\beta_{\text{CQ}}/\beta_{\text{RP},k}) \leq \epsilon_1] \rightarrow 1$  as  $(n, p) \rightarrow \infty$ .*

**Remarks.** The case of  $\epsilon_1 = 1$  serves as the reference for equal asymptotic performance, with values  $\epsilon_1 < 1$  corresponding to the  $\bar{T}_k^2$  statistic being asymptotically more powerful than the test of CQ. To interpret the result, note that Jensen’s inequality implies that the ratio  $\text{tr}(\Sigma)^2 / \|\Sigma\|_F^2$  lies between 1 and  $p$ , for any choice of  $\Sigma$ . As such, it is reasonable to interpret this ratio as a measure of the *effective dimension* of the covariance structure (c.f. Examples 1 and 2 below).<sup>3</sup> The message of Theorem 3 is that as long as the sample size  $n$  grows faster than the effective dimension (with  $\delta$  being uniformly oriented), then our projection-based test is asymptotically superior to the test of CQ.

The ratio  $\text{tr}(\Sigma)^2 / \|\Sigma\|_F^2$  can also be viewed as measuring the *decay rate* of the spectrum of  $\Sigma$ , with the condition  $\text{tr}(\Sigma)^2 / \|\Sigma\|_F^2 \ll p$  indicating rapid decay. This condition means that the data has low variance in “most” directions in  $\mathbb{R}^p$ , and so projecting onto a random set of  $k$  directions will likely map the data into a low-variance subspace in which it is harder for chance variation to explain away the correct hypothesis, thereby resulting in greater power.

**Example 1.** One instance of spectrum decay occurs when the top, say  $s$ , eigenvalues of  $\Sigma$  contain most of the mass in the spectrum. When  $\Sigma$  is diagonal, this has the interpretation that  $s$  variables capture most of the total variance in the data. For simplicity, assume  $\lambda_1 = \dots = \lambda_s > 1$  and  $\lambda_{s+1} = \dots = \lambda_p = 1$ , which is similar to the *spiked covariance model* introduced by Johnstone [20]. If the top  $s$  eigenvalues contain half of the total mass of the spectrum, then  $s \lambda_1 = (p - s)$ , and a simple calculation shows that

$$\frac{\text{tr}(\Sigma)^2}{\|\Sigma\|_F^2} = \frac{4 \lambda_1^2}{\lambda_1^2 + \lambda_1} s \leq 4s. \quad (14)$$

This illustrates the idea that condition (13) is satisfied as long as  $n$  grows at a faster rate than the effective number of variables  $s$ . It is straightforward to check that this example satisfies the conditions of (A5) of Theorem 3 when, for instance,  $\lambda_1 = o(k)$ .

**Example 2.** Another example of spectrum decay can be specified by  $\lambda_i(\Sigma) \propto i^{-\nu}$ , for some absolute proportionality constant, a rate parameter  $\nu \in (0, \infty)$ , and  $i = 1, \dots, p$ . This type of decay occurs in various sparse signal models, and arises in connection with the Fourier coefficients of functions

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<sup>3</sup>This ratio has also been studied as an effective measure of matrix rank in the context of low-rank matrix reconstruction [19].

in Sobolev ellipsoids [21] [22§7.2]. Noting that  $\text{tr}(\Sigma) \asymp \int_1^p x^{-\nu} dx$  and  $\|\Sigma\|_F^2 \asymp \int_1^p x^{-2\nu} dx$ , direct computation of the integrals shows that

$$\frac{\text{tr}(\Sigma)^2}{\|\Sigma\|_F^2} \asymp \begin{cases} 1 & \text{if } \nu > 1 \\ \log^2 p & \text{if } \nu = 1 \\ p^{2(1-\nu)} & \text{if } \nu \in (\frac{1}{2}, 1) \\ p/\log p & \text{if } \nu = \frac{1}{2} \\ p & \text{if } \nu \in (0, \frac{1}{2}) \end{cases}.$$

Thus, a decay rate given by  $\nu \geq 1$  is easily sufficient for condition (13) to hold unless the dimension grows exponentially with  $n$ . On the other hand, decay rates associated to  $\nu \leq 1/2$  are too slow for condition (13) to hold when  $n \ll p$ , and rates corresponding to  $\nu \in (\frac{1}{2}, 1)$  lead to a more nuanced competition between  $p$  and  $n$ . We also point out that the assumptions of Theorem 3 hold for all  $\nu \in (0, 1]$ , when the dimension  $p$  must satisfies  $\log p = o(k)$ .

The proof of Theorem 3 is a direct application of Propositions 1 and 2.

*Proof of Theorem 3.* Recalling  $\text{ARE}(\beta_{\text{CQ}}/\beta_{\text{RP},k}) = \left( \frac{n\|\delta\|_2^2}{\|\Sigma\|_F^2} / \sqrt{n\Delta_k} \right)^2$ , with  $k = \lfloor n/2 \rfloor$  and  $y = 1/2$ , the event of interest,

$$\text{ARE}(\beta_{\text{CQ}}/\beta_{\text{RP},k}) \leq \epsilon_1, \quad (15)$$

is the same as

$$\frac{n}{\|\Sigma\|_F^2} \frac{1}{\epsilon_1} \leq \left( \frac{\overline{\Delta}_k}{\|\delta\|_2^2} \right)^2.$$

By Propositions 1 or 2, we know that for any  $c < 1$ , the probability of the event

$$\frac{ck}{\text{tr}(\Sigma)} \leq \frac{\overline{\Delta}_k}{\|\delta\|_2^2} \quad (16)$$

tends to 1 as  $(n, p) \rightarrow \infty$ . Consequently, as long as the inequality

$$\frac{n}{\|\Sigma\|_F^2} \frac{1}{\epsilon_1} \leq \left( \frac{ck}{\text{tr}(\Sigma)} \right)^2 \quad (17)$$

holds for all large  $n$ , then the event (15) of interest will also have probability tending to 1. Replacing  $k$  with  $\frac{n}{2} \cdot [1 - o(1)]$ , the last condition is the same as

$$n \geq \frac{\text{tr}(\Sigma)^2}{\|\Sigma\|_F^2} \cdot \frac{4}{\epsilon_1 c^2 [1 - o(1)]^2}. \quad (18)$$

□

### 3.5 Power comparison with SD

We now turn to a comparison of the asymptotic power of our test with that of Srivastava and Du (SD) [6, 7]. Recall that  $R$  denotes the correlation matrix.

**Theorem 4.** Fix a number  $\epsilon_1 > 0$ , and let  $c_1(\epsilon_1)$  be any constant strictly greater than  $\frac{4}{\epsilon_1}$ .

(i) (Uniform shift). Assume  $\|D_\sigma^{-1}\|_{op}/\text{tr}(D_\sigma^{-1}) = o(1)$ , and that the conditions of Proposition 1[(i)] hold. Then, as long as the inequality

$$n \geq c_1(\epsilon_1) \left( \frac{\text{tr}(\Sigma)}{p} \right)^2 \left( \frac{\text{tr}(D_\sigma^{-1})}{\|R\|_F} \right)^2 \quad (19)$$

is satisfied for all large  $n$ , we have the limit  $\mathbb{P}[\text{ARE}(\beta_{\text{SD}}/\beta_{\text{RP},k}) \leq \epsilon_1] \rightarrow 1$  as  $(n, p) \rightarrow \infty$ .

(ii) (Tilted shift). Assume  $\|R\|_{op}/\text{tr}(R) = o(1)$ , and that the conditions of Proposition 2 hold. Then, as long as the inequality

$$n \geq c_1(\epsilon_1) \left( \frac{\text{tr}(R)}{\|R\|_F} \right)^2 \quad (20)$$

is satisfied for all large  $n$ , we have the limit  $\mathbb{P}[\text{ARE}(\beta_{\text{SD}}/\beta_{\text{RP},k}) \leq \epsilon_1] \rightarrow 1$  as  $(n, p) \rightarrow \infty$ .

**Remarks.** Unlike the comparison with the CQ test, the correlation matrix  $R$  plays a large role in determining the relative efficiency between our procedure and the SD test. The correlation matrix enters in two different ways. First, the Frobenius norm  $\|R\|_F$  is larger when the data variables are more correlated. Second, if  $\Sigma$  has a large number of small eigenvalues, then  $\text{tr}(D_\sigma^{-1})$  is very large when the variables are uncorrelated, i.e. when  $\Sigma$  is diagonal. Letting  $U\Lambda U^\top$  be a spectral decomposition of  $\Sigma$ , with  $u_i$  being the  $i$ th column of  $U^\top$ , note that  $(D_\sigma)_{ii} = u_i^\top \Lambda u_i$ . When the data variables are correlated, the vector  $u_i$  will typically have many nonzero components, which will give  $(D_\sigma)_{ii}$  a contribution from some of the larger eigenvalues of  $\Sigma$ , and prevent  $(D_\sigma)_{ii}$  from being too small. For example, if  $u_i$  is uniformly distributed on the unit sphere, as in Example 4 below, then on average  $\mathbb{E}[(D_\sigma)_{ii}] = \text{tr}(\Sigma)/p$ . Therefore, correlation has the effect of mitigating the growth of  $\text{tr}(D_\sigma^{-1})$ . Since the SD test statistic [6] can be thought of as a version of the Hotelling  $T^2$  with a diagonal estimator of  $\Sigma$ , the SD test statistic makes no essential use of correlation structure. By contrast, our  $\bar{T}_k^2$  statistic *does* take correlation into account, and so it is understandable that correlated data enhance the performance of our test relative to SD.

**Example 3.** Suppose the correlation matrix  $R \in \mathbb{R}^{p \times p}$  has a block-diagonal structure, with  $m$  identical blocks  $B \in \mathbb{R}^{d \times d}$  along the diagonal:

$$R = \begin{pmatrix} B & & \\ & \ddots & \\ & & B \end{pmatrix}. \quad (21)$$

Note that  $p = m \cdot d$ . Fix a number  $\rho \in (-1, 1)$ , and let  $B$  have diagonal entries equal to 1, and off-diagonal entries equal to  $\rho$ , i.e.  $B = (1 - \rho)I_{d \times d} + \rho \mathbf{1}\mathbf{1}^\top$ , where  $\mathbf{1} \in \mathbb{R}^d$  is the all-ones vector. Consequently,  $R$  is positive-definite, and we may consider  $\Sigma = R$  for simplicity. Since  $\|B\|_F^2 = d + 2\rho^2 \binom{d}{2}$ , and  $\|R\|_F^2 = m \|B\|_F^2$ , it follows that

$$\|R\|_F^2 = [1 + \rho^2(d - 1)]p.$$

Also, in this example we have  $\text{tr}(\Sigma) = \text{tr}(D_\sigma^{-1}) = p$  and  $p/d = m$ , which implies

$$\left( \frac{\text{tr}(\Sigma)}{p} \right)^2 \left( \frac{\text{tr}(D_\sigma^{-1})}{\|R\|_F} \right)^2 = \frac{p}{1 + \rho^2(d - 1)} \leq \frac{m}{\rho^2}. \quad (22)$$

Similarly,

$$\left(\frac{\text{tr}(R)}{\|R\|_F}\right)^2 \leq \frac{m}{\rho^2}.$$

Consequently, we conclude that the sufficient conditions (19) and (20) from Theorem 4 are satisfied when  $n$  grows at a faster rate than the number of blocks  $m$ . Note too that the spectrum of  $\Sigma$  consists of  $m$  copies of  $\lambda_{\max}(\Sigma) = (1 - \rho) + \rho d$  and  $(p - m)$  copies of  $\lambda_{\min}(\Sigma) = 1 - \rho$ , which means when  $\rho$  is not too small, the number of blocks is the same as the number of dominant eigenvalues—revealing a parallel with Example 1. From these observations, it is straightforward to check that this example satisfies the assumptions of Theorem 4. The simulations in Section 4.1 give an example where  $R$  has the form (21), and the variables corresponding to each block are highly correlated.

**Example 4.** To consider the performance of our test in a case where  $\Sigma$  is not constructed deterministically, Section 4.1 illustrates simulations involving *randomly selected* matrices  $\Sigma$  for which the statistic  $\bar{T}_k^2$  is more powerful than the tests of BS, CQ, and SD. Random correlation structure can be generated by sampling the matrix of eigenvectors of  $\Sigma$  from the uniform (Haar) distribution on the orthogonal group, and then imposing various decay constraints on the eigenvalues of  $\Sigma$ . Additional details are provided in Section 4.1.

**Example 5.** It is possible to show that the sufficient conditions (19) and (20) *require* non-trivial correlation in the high-dimensional setting. To see this, consider an example where the data are completely free of correlation, i.e., where  $R = I_{p \times p}$ . Then,  $\|R\|_F = \sqrt{p}$ , and Jensen's inequality implies that  $\text{tr}(D_\sigma^{-1}) \geq p^2 / \text{tr}(D_\sigma) = p^2 / \text{tr}(\Sigma)$ , giving  $\left(\frac{\text{tr}(\Sigma)}{p}\right)^2 \left(\frac{\text{tr}(D_\sigma^{-1})}{\|R\|_F}\right)^2 \geq p$ . Altogether, this shows if the data exhibit very low correlation, then condition (19) cannot hold when  $p$  grows faster than  $n$ . This is confirmed by the simulations of Section 4.1 involving diagonal covariance matrices.

We now turn to the proof of Theorem 4, which makes use of the following concentration bounds for Gaussian quadratic forms.

**Lemma 1.** *Let  $A \in \mathbb{R}^{p \times p}$  be a positive semidefinite matrix with  $\|A\|_{op} > 0$ , and let  $Z \sim N(0, I_{p \times p})$ . Then, for any  $t > 0$ ,*

$$\mathbb{P} \left[ \frac{Z^\top A Z}{\text{tr}(A)} \geq \left( 1 + t \sqrt{\frac{\|A\|_{op}}{\text{tr}(A)}} \right)^2 \right] \leq \exp(-t^2/2), \quad (23)$$

and for any  $t \in (0, \sqrt{\frac{\text{tr}(A)}{\|A\|_{op}} - 1})$ , we have

$$\mathbb{P} \left[ \frac{Z^\top A Z}{\text{tr}(A)} \leq \left( \sqrt{1 - \frac{\|A\|_{op}}{\text{tr}(A)}} - t \sqrt{\frac{\|A\|_{op}}{\text{tr}(A)}} \right)^2 \right] \leq \exp(-t^2/2). \quad (24)$$

See Appendix A for a short proof of this result. These bounds are similar to results in the papers of Bechar and Laurent and Massart [23, 24], but have error terms involving the operator norm as opposed to the Frobenius norm, and hence may be of independent interest.

Equipped with this lemma, we can now prove Theorem 4.

*Proof of Theorem 4.* We only give the proof under the conditions of Proposition 2, since the proof is essentially the same under the conditions of Proposition 1[(i)]. Let us define the event of interest  $\mathcal{E}_n := \{\text{ARE}(\beta_{\text{SD}}/\beta_{\text{RP},k}) \leq \epsilon_1\}$ , where we recall  $\text{ARE}(\beta_{\text{SD}}/\beta_{\text{RP},k}) = \left(\frac{n\delta^\top D_\sigma^{-1}\delta}{\|R\|_F^2} / \sqrt{n}\Delta_k\right)^2$ , with  $k = \lfloor n/2 \rfloor$  and  $y = 1/2$ . The event  $\mathcal{E}_n$  holds if and only if

$$\frac{n}{\|R\|_F^2} \frac{1}{\epsilon_1} \leq \left(\frac{\overline{\Delta}_k}{\|\delta\|_2^2}\right)^2 \left(\frac{\|\delta\|_2^2}{\delta^\top D_\sigma^{-1}\delta}\right)^2. \quad (25)$$

We consider the two factors on the right hand side of (25) separately. By Proposition 2, the first factor  $\frac{\overline{\Delta}_k}{\|\delta\|_2^2}$  satisfies

$$\frac{\overline{\Delta}_k}{\|\delta\|_2^2} \bigg/ \frac{k}{\text{tr}(\Sigma)} \rightarrow 1 \quad \text{in probability under } \mathbb{P}_\delta. \quad (26)$$

Since the second factor  $\frac{\|\delta\|_2^2}{\delta^\top D_\sigma^{-1}\delta}$  in line (25) is invariant under scaling of  $\delta$ , we may write  $\delta = \Sigma^{1/2}Z$  with  $Z \sim N(0, I_{p \times p})$ . Next, using Lemma 1, and  $\text{tr}(\Sigma^{1/2}D_\sigma^{-1}\Sigma^{1/2}) = \text{tr}(R)$ , we have

$$\frac{\|\delta\|_2^2}{\delta^\top D_\sigma^{-1}\delta} \bigg/ \frac{\text{tr}(\Sigma)}{\text{tr}(R)} \rightarrow 1 \quad \text{in probability under } \mathbb{P}_\delta, \quad (27)$$

since  $\frac{\|\delta\|_2^2}{\delta^\top D_\sigma^{-1}\delta} \cdot \frac{\text{tr}(R)}{\text{tr}(\Sigma)} = \frac{Z^\top \Sigma Z}{Z^\top \Sigma^{1/2} D_\sigma^{-1} \Sigma^{1/2} Z} \cdot \frac{\text{tr}(R)}{\text{tr}(\Sigma)} \rightarrow 1$  in probability under  $\mathbb{P}_\delta$ . Consequently, for any  $c \in (0, 1)$ , we combine (26) and (27) to conclude that  $\mathbb{P}(\mathcal{E}_n) \rightarrow 1$  as long as the inequality

$$\frac{n}{\|R\|_F^2} \frac{1}{\epsilon_1} \leq c \left(\frac{k}{\text{tr}(\Sigma)}\right)^2 \left(\frac{\text{tr}(\Sigma)}{\text{tr}(R)}\right)^2 \quad (28)$$

holds for all large  $n$ . Replacing  $k$  with  $\frac{n}{2} \cdot [1 - o(1)]$ , the last condition is equivalent to

$$n \geq \left(\frac{\text{tr}(R)}{\|R\|_F}\right)^2 \cdot \frac{4}{\epsilon_1 c [1 - o(1)]^2}. \quad (29)$$

□

## 4 Comparisons with simulated data

In this section, we compare our procedure to a broad collection of competing methods on simulated data, illustrating the effects of the different factors involved in Theorems 3 and 4.

### 4.1 Description of models

We generated ROC curves using multivariate normal data, sampled under 10 different choices of the parameters  $\delta$  and  $\Sigma$ . For each ROC curve, we sampled  $n_1 = n_2 = 50$  data points from each of the distributions  $N(\mu_1, \Sigma)$  and  $N(\mu_2, \Sigma)$  in  $p = 200$  dimensions, and repeated the process 500 times with  $\delta = \mu_1 - \mu_2 = 0$  under  $\mathbf{H}_0$ , and 500 times with  $\delta \neq 0$  under  $\mathbf{H}_1$ . Hence, each ROC curve reflects the results of 1000 two-sample problems. We organize the 10 parameter settings according to 5 different choices of  $\Sigma$ , and 2 different choices of  $\delta$ , as displayed in the following table.

Setting #	Choice of $\Sigma$					Choice of $\delta$	
	diagonal	random	slow decay	fast decay	block structure	uniform	tilted
1	X		X			X	
2	X			X		X	
3		X	X			X	
4		X		X		X	
5					X	X	
6	X		X				X
7	X			X			X
8		X	X				X
9		X		X			X
10					X		X

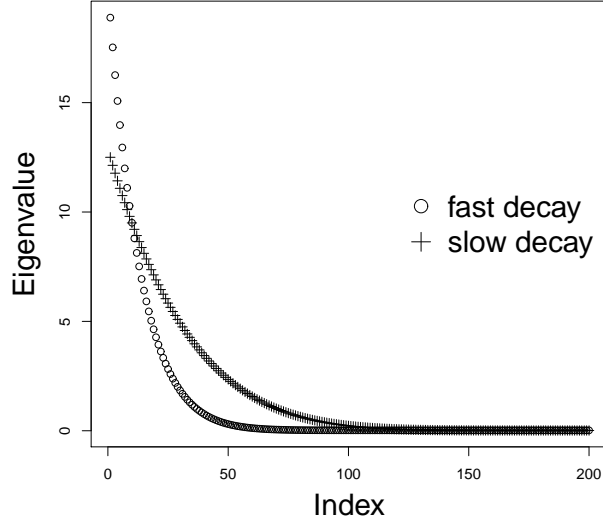
For each two-sample problem under a *uniform* shift, the vector  $\delta$  was drawn independently as  $Z/\|Z\|_2$  with  $Z \sim N(0, I_{p \times p})$ , and for each problem under a *tilted* shift, the vector  $\delta$  was drawn independently as  $2\Sigma^{1/2}Z/\|\Sigma^{1/2}Z\|_2$ , so as to conform to the conditions in Theorems 3 and 4. Similarly, for cases where  $\Sigma$  was drawn randomly, an independent copy was drawn for each two-sample problem, as described below.

To describe the 5 choices of  $\Sigma$ , let  $U\Lambda U^\top$  denote a spectral decomposition of  $\Sigma$ . A *diagonal* choice of  $\Sigma$  corresponds to  $U = I_{p \times p}$ , and a *random* choice of  $\Sigma$  corresponds to drawing  $U$  from the uniform (Haar) distribution on the orthogonal group (see the paper [25] for sampling details). Note that  $U = I_{p \times p}$  results in independent variables, whereas a random  $U$  leads to correlated variables. To consider two rates of spectral decay, which we refer to as *slow* and *fast*, we selected  $p$  equally spaced eigenvalues  $\lambda_1, \dots, \lambda_p$  between 0.01 and 1, and raised them to the power 15 for fast decay, and the power 6 for slow decay. We then added 0.001 to each eigenvalue to control the condition number of  $\Sigma$ , and rescaled them so that  $\|\Sigma\|_F = \sqrt{\lambda_1^2 + \dots + \lambda_p^2} = 50$  (fixing a common amount of variance in both the slow and fast cases). Plots of the resulting spectra are shown in Figure 2. The last setting involves a sparse matrix  $\Sigma$  with 40 small groups of highly correlated variables, and we refer to this as the *block-structured* case. This case does not depend on the previously described choices of  $U$  or  $\Lambda$ . Specifically, the block-structured choice of  $\Sigma$  was constructed using 40 identical blocks  $B \in \mathbb{R}^{5 \times 5}$  along the diagonal, with the diagonal entries of  $B$  equal to 1, and the off-diagonal entries of  $B$  equal to  $\rho := 0.8$  (c.f. Example 3).

## 4.2 ROC curve comparisons

In addition to our random projection (RP)-based test, we implemented the methods of BS [5], SD [6], and CQ [8], which are all designed specifically for problem (1) in the high-dimensional setting. For the sake of completeness, we also show comparisons with two recent non-parametric procedures that are based on kernel methods: maximum mean discrepancy (MMD) [13], and kernel Fisher discriminant analysis (KFDA) [14], as well as a test based on area-under-curve maximization, denoted TreeRank [9]. Note that the curves labeled RP-1, RP-30, and RP-100 correspond to computing the statistic  $\bar{T}_k^2$  by averaging over 1, 30, and 100 matrices  $P_k$  with i.i.d.  $N(0, 1)$  entries.

Overall, the ROC curves in Figures 3 and 4 show that the RP and SD tests perform the best within this collection of procedures, with RP test having an advantage for correlated variables



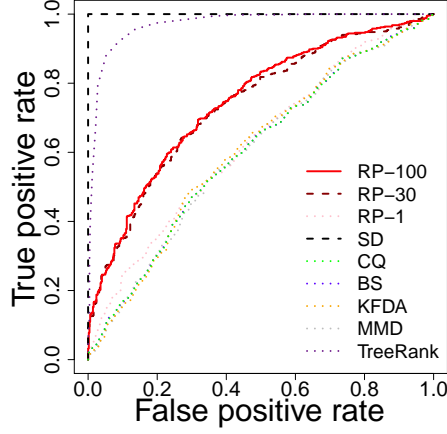
**Figure 2.** Plots of two sets of eigenvalues  $\lambda_1, \dots, \lambda_p$ , with slow and fast decay, both satisfying  $\|\Sigma\|_F = \sqrt{\lambda_1^2 + \dots + \lambda_p^2} = 50$ . To interpret the number of non-negligible eigenvalues, there are 29 eigenvalues greater than a tenth of  $\lambda_{\max}(\Sigma)$  in the case of fast decay, and there are 65 eigenvalues greater than a tenth of  $\lambda_{\max}(\Sigma)$  in the case of slow decay.

and the SD test having an advantage for independent variables. Furthermore, the plots indicate that essentially no additional power is gained by averaging the  $\bar{T}_k^2$  statistic with more than 30 projections.

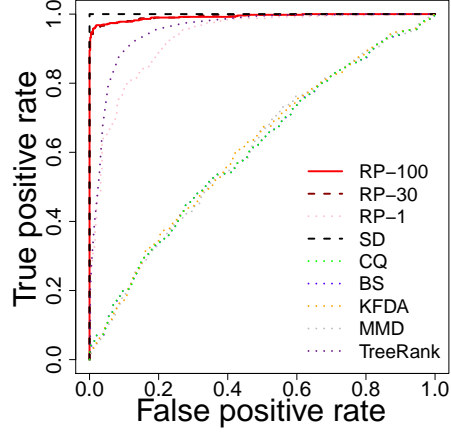
On a qualitative level, Figures 3 and 4 reveal some striking differences between our procedure and the competing tests. Comparing independent variables versus correlated variables, i.e. panels (a) and (b), with panels (c) and (d) in both figures, we see that the tests of SD and TreeRank lose power in the presence of correlated data. Meanwhile, the ROC curve of our test is essentially unchanged when passing from independent variables to correlated variables. Similarly, our test also exhibits an advantage when the correlation structure is prescribed in a block-diagonal manner in panel (e). The agreement of this effect with Theorem 4 is explained in the remarks and examples after that theorem. Comparing slow spectral decay versus fast spectral decay, i.e. panels (a) and (c), with panels (b) and (d), we see that the competing tests are essentially insensitive to the change in spectrum, whereas our test is able to take advantage of low-dimensional covariance structure. The remarks and examples of Theorem 3 give a theoretical justification for this observation. To offer a more quantitative assessment of the ROC curves in terms of Theorems 3 and 4, Table 1 below summarizes approximate values of the ARE-determining quantities in those results.

The fluctuations of 100 p-values calculated from a single dataset by averaging over different numbers of projections is illustrated in Figure 5 below. The dataset was drawn according to Setting 4. In particular, we see that the order of magnitude of the p-values is stable when 100 projections are used for averaging. When the number of projections is increased to  $10^4$ , all 100 p-values lie strictly between 0.022 and 0.029. We remark that the  $\bar{T}_k^2$  statistic can be computed by averaging over  $10^4$  projections in about 30 seconds using R code on a basic laptop.

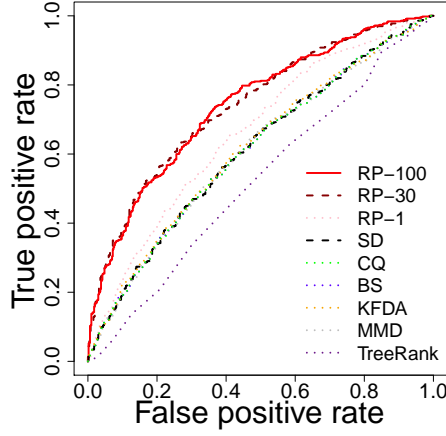




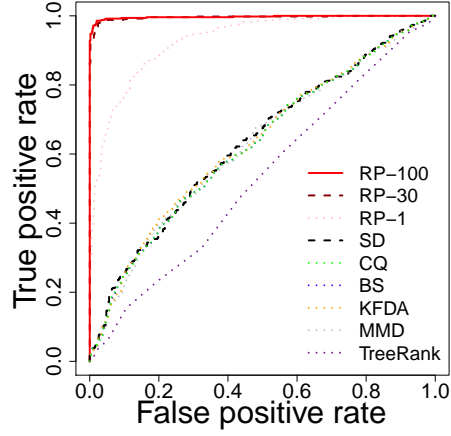
(a) Setting (1): diagonal  $\Sigma$ , slow decay



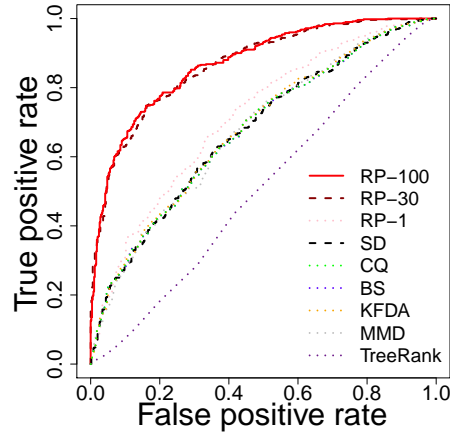
(b) Setting (2): diagonal  $\Sigma$ , fast decay



(c) Setting (3): random  $\Sigma$ , slow decay

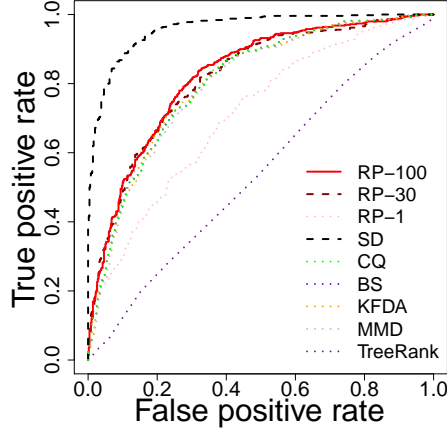


(d) Setting (4): random  $\Sigma$ , fast decay

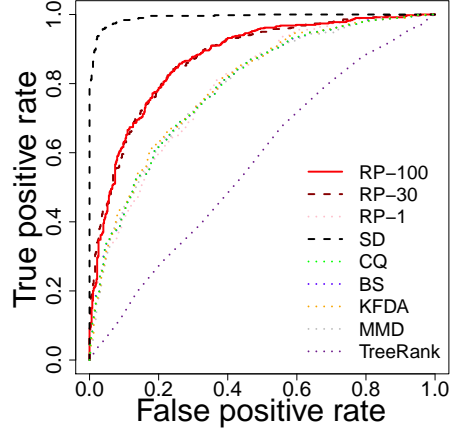


(e) Setting (5): block covariance

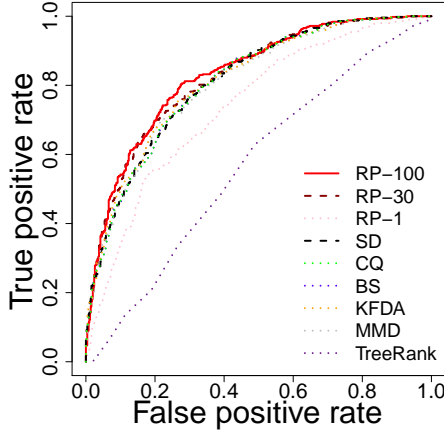
**Figure 3.** (Uniform shift) ROC curves of several test statistics for five different settings of  $\Sigma$  with the shift vector  $\delta$  drawn uniformly from the unit sphere: (1) Diagonal covariance / slow decay, (2) Diagonal covariance / fast decay, (3) Random covariance / slow decay, and (4) Random covariance / fast decay, (5) Block covariance.



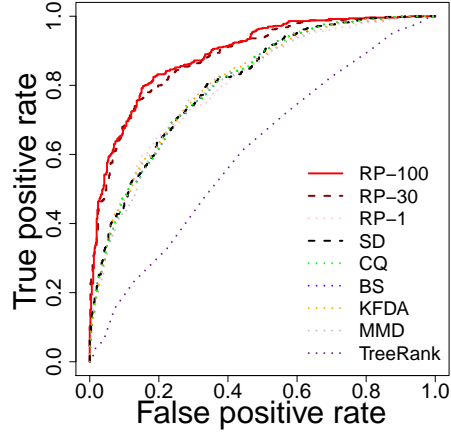
(a) Setting (6): diagonal  $\Sigma$ , slow decay



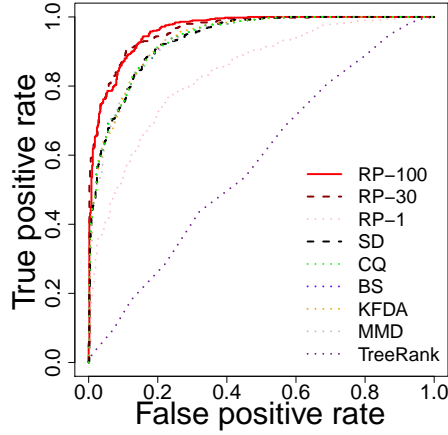
(b) Setting (7): diagonal  $\Sigma$ , fast decay



(c) Setting (8): random  $\Sigma$ , slow decay



(d) Setting (9) random  $\Sigma$ , fast decay

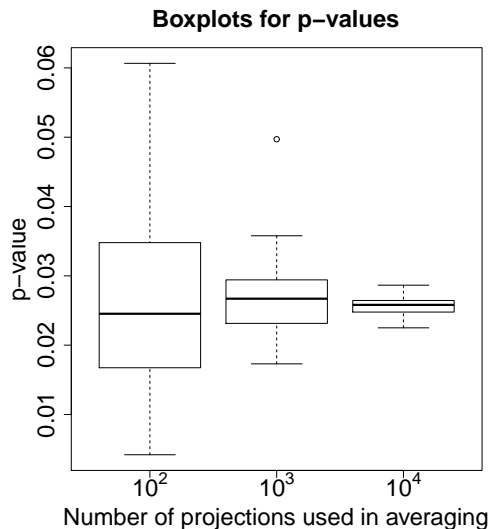


(e) Setting (10): block covariance

**Figure 4.** (Tilted shift) ROC curves of several test statistics for five different settings of  $\Sigma$  with the shift vector  $\delta$  drawn from  $N(0, s\Sigma)$  with  $s = \sqrt{\text{tr}(\Sigma)}/2$ : (6) Diagonal covariance / slow decay, (7) Diagonal covariance / fast decay, (8) Random covariance / slow decay, and (9) Random covariance / fast decay, (10) Block covariance.

Quantity	diagonal $\Sigma$ , slow decay	diagonal $\Sigma$ , fast decay	random $\Sigma$ , slow decay	random $\Sigma$ , fast decay	block structure
$\text{tr}(\Sigma)^2 / \ \Sigma\ _F^2$	54	25	54	25	56
$\left(\frac{\text{tr}(\Sigma)}{p}\right)^2 \left(\frac{\text{tr}(D_\sigma^{-1})}{\ R\ _F}\right)^2$	$4.6 \times 10^5$	$3.5 \times 10^5$	58	30	56
$\text{tr}(R)^2 / \ R\ _F^2$	200	200	55	26	56

**Table 1.** Approximate values of the quantities  $\text{tr}(\Sigma)^2 / \|\Sigma\|_F^2$ ,  $\left(\frac{\text{tr}(\Sigma)}{p}\right)^2 \left(\frac{\text{tr}(D_\sigma^{-1})}{\|R\|_F}\right)^2$ , and  $\text{tr}(R)^2 / \|R\|_F^2$ , for the five choices of  $\Sigma$  in the simulated data experiments. For cases involving a randomly drawn  $\Sigma$ , the quantities are obtained as the rounded average of 500 draws. Theorems 3 and 4 assert that these quantities determine the relative performance of our test with CQ and SD respectively. The value of  $\left(\frac{\text{tr}(\Sigma)}{p}\right)^2 \left(\frac{\text{tr}(D_\sigma^{-1})}{\|R\|_F}\right)^2$  is very large in the case of diagonal  $\Sigma$ , since  $\text{tr}(D_\sigma^{-1})$  involves reciprocals of many small eigenvalues. This effect disappears once correlation is introduced.



**Figure 5.** Using a single dataset drawn according to the conditions of Setting 4, we multiplied the shift  $\delta$  by a factor of 35 to bring the p-values into an interesting range. For each choice of  $N = 10^2, 10^3, 10^4$ , we calculated 100 p-values with the statistic  $\bar{T}_k^2$  by averaging over  $N$  projections.

## 5 Conclusion

We have proposed a novel testing procedure for the two-sample test of means in high dimensions. This procedure can be implemented in a simple manner by averaging over projected versions of the classical Hotelling  $T^2$  test. In addition to deriving the asymptotic power function for this test, we have provided interpretable conditions on the covariance and correlation matrices for achieving greater power than competing tests in the sense of asymptotic relative efficiency. Specifically, our theoretical comparisons show that our test is well-suited to interesting regimes where the data variables are correlated, or where most of the variance can be captured in a small number of variables. Furthermore, in the realistic case of  $(n, p) = (98, 200)$ , these types of conditions were shown to correspond to favorable performance of our test with several competitors in ROC curve

comparisons on simulated data. Extensions of this work may include more refined applications of random projection to other high-dimensional testing problems.

**Acknowledgements.** The authors thank Sandrine Dudoit and Peter Bickel for helpful discussions. Art Owen is thanked for suggesting the case  $\delta \sim N(0, \Sigma)$ . MEL gratefully acknowledges the support of the DOE CSGF Fellowship, under grant number DE-FG02-97ER25308, and LJ the support of Stand Up to Cancer. MJW and MEL were partially supported by NSF grant DMS-0907632.

## A Proofs of scaling results for $\overline{\Delta}_k$ (Propositions 1 and 2)

Before proceeding to the proofs of Propositions 1 and 2, we prove Lemma 1, which was stated in the main text. We also state and prove Lemmas 2 and 3, concerning the eigenvalues of  $\mathbb{E}_{P_k}[P_k(P_k^\top \Sigma P_k)^{-1}P_k^\top]$ .

**Proof of Lemma 1.** Note that the function  $f(Z) := \sqrt{Z^\top A Z} = \|A^{1/2}Z\|_2$  has Lipschitz constant  $\sqrt{\|A\|_{\text{op}}}$  with respect to the Euclidean norm on  $\mathbb{R}^p$ . By the Cirel'son-Ibragimov-Sudakov inequality for Lipschitz functions of Gaussian vectors [26], we have for any  $s > 0$ ,

$$\mathbb{P}[f(Z) \leq \mathbb{E}[f(Z)] - s] \leq \exp\left(\frac{-s^2}{2\|A\|_{\text{op}}}\right). \quad (30)$$

From the Poincaré inequality for Gaussian measures [27], the variance of  $f(Z)$  is bounded above as  $\text{var}[f(Z)] \leq \|A\|_{\text{op}}$ . Noting that  $\mathbb{E}[f(Z)^2] = \text{tr}(A)$ , we see that the expectation of  $f(Z)$  is lower bounded as

$$\mathbb{E}[f(Z)] \geq \sqrt{\text{tr}(A) - \|A\|_{\text{op}}}.$$

Substituting this lower bound into the concentration inequality (30) yields

$$\mathbb{P}\left[f(Z) \leq \sqrt{\text{tr}(A) - \|A\|_{\text{op}}} - s\right] \leq \exp\left(\frac{-s^2}{2\|A\|_{\text{op}}}\right).$$

Finally, letting  $t \in \left(0, \sqrt{\frac{\text{tr}(A)}{\|A\|_{\text{op}}}} - 1\right)$ , and choosing  $s^2 = t^2 \|A\|_{\text{op}}$  yields the claim (24).

To obtain the second bound (24) in Lemma 1, we use the upper-tail version of the inequality (30), namely  $\mathbb{P}[f(Z) \geq \mathbb{E}[f(Z)] + s] \leq \exp\left(\frac{-s^2}{2\|A\|_{\text{op}}}\right)$  for  $s > 0$ . By Jensen's inequality, we have

$$\mathbb{E}[f(Z)] = \mathbb{E}\sqrt{Z^\top A Z} \leq \sqrt{\mathbb{E}[Z^\top A Z]} = \sqrt{\text{tr}(A)},$$

from which we obtain  $\mathbb{P}\left[f(Z) \geq \sqrt{\text{tr}(A)} + s\right] \leq \exp\left(\frac{-s^2}{2\|A\|_{\text{op}}}\right)$ , and setting  $s^2 = t^2 \|A\|_{\text{op}}$  for  $t > 0$  yields the claim (23).  $\square$

**Lemma 2.** For any positive definite matrix  $\Sigma \in \mathbb{R}^{p \times p}$ , we have the upper bound

$$\left\|\mathbb{E}_{P_k}[P_k(P_k^\top \Sigma P_k)^{-1}P_k^\top]\right\|_{\text{op}} \leq \frac{1}{\lambda_{\min}(\Sigma)}.$$

*Proof.* Using Jensen's inequality, and the fact that the matrix  $P_k(P_k^\top \Sigma P_k)^{-1}P_k^\top$  has the same non-zero eigenvalues as the matrix  $(P_k^\top \Sigma P_k)^{-1}P_k^\top P_k = (P_k^\top \Sigma P_k)^{-1}$ , we have

$$\left\| \mathbb{E}_{P_k}[P_k(P_k^\top \Sigma P_k)^{-1}P_k^\top] \right\|_{\text{op}} \leq \mathbb{E}_{P_k} \left\| (P_k^\top \Sigma P_k)^{-1} \right\|_{\text{op}}. \quad (31)$$

Since  $\left\| (P_k^\top \Sigma P_k)^{-1} \right\|_{\text{op}}$  is the reciprocal of  $\lambda_{\min}(P_k^\top \Sigma P_k)$ , we use the variational characterization of the minimum eigenvalue to obtain the lower bound,

$$\begin{aligned} \lambda_{\min}(P_k^\top \Sigma P_k) &= \inf_{\|x\|_2=1} x^\top P_k^\top \Sigma P_k x \\ &\geq \inf_{\|y\|_2=1} y^\top \Sigma y \cdot \inf_{\|x\|_2=1} \|P_k x\|_2^2 \\ &= \lambda_{\min}(\Sigma) \cdot 1. \end{aligned} \quad (32)$$

□

**Lemma 3.** For any positive definite matrix  $\Sigma \in \mathbb{R}^{p \times p}$ , we have the lower bound

$$\text{tr}(\mathbb{E}_{P_k}[P_k(P_k^\top \Sigma P_k)^{-1}P_k^\top]) \geq k \cdot \frac{p}{\text{tr}(\Sigma)}.$$

*Proof.* Combining the cyclic property of trace with the identity  $P_k^\top P_k = I_{k \times k}$  yields

$$\text{tr}(\mathbb{E}_{P_k}[P_k(P_k^\top \Sigma P_k)^{-1}P_k^\top]) = \mathbb{E}_{P_k}[\text{tr}(P_k^\top \Sigma P_k)^{-1}].$$

Letting  $\{\lambda_i\}_{i=1}^k$  denote the eigenvalues of  $P_k^\top \Sigma P_k$ , we then have

$$\frac{1}{k} \text{tr}(P_k^\top \Sigma P_k)^{-1} = \frac{1}{k} \sum_{i=1}^k \frac{1}{\lambda_i} \geq \frac{k}{\sum_{i=1}^k \lambda_i},$$

where we have applied Jensen's inequality to the function  $f(t) = 1/t$ , which is convex on the positive real line. Applying the same argument with the expectation  $\mathbb{E}_{P_k}$ , we obtain

$$\frac{1}{k} \text{tr}(\mathbb{E}_{P_k}[P_k(P_k^\top \Sigma P_k)^{-1}P_k^\top]) \geq \mathbb{E}_{P_k}[k / \text{tr}(P_k^\top \Sigma P_k)] \geq \frac{k}{\mathbb{E}_{P_k}[\text{tr}(P_k^\top \Sigma P_k)]}. \quad (33)$$

Let  $\Sigma = U \Lambda U^\top$  be a spectral decomposition of  $\Sigma$ . Since  $P_k$  follows a unitarily invariant distribution,  $P_k^\top \Sigma P_k = P_k^\top U \Lambda U^\top P_k$ , and the  $i$ th diagonal entry of this matrix can be written as  $u_i^\top \Lambda u_i$ , where  $u_i$  is the  $i$ th column of  $P_k$ . Since  $u_i$  is uniformly distributed on the unit sphere of  $\mathbb{R}^p$ , the expected value of the square of any coordinate of  $u_i$  is equal to  $1/p$ . Hence,  $\mathbb{E}[u_i^\top \Lambda u_i] = \text{tr}(\Lambda)/p$ , and so  $\mathbb{E}_{P_k}[\text{tr}(P_k^\top \Sigma P_k)] = k \text{tr}(\Sigma)/p$ . □

## A.1 Proof of Proposition 1.

*Proof of part (i).* We may represent  $\delta/\|\delta\|_2$  as  $Z/\|Z\|_2$  for a standard  $p$ -dimensional Gaussian vector  $Z$ . Let  $B = \mathbb{E}_{P_k}[P_k(P_k^\top \Sigma P_k)^{-1}P_k^\top]$  so that

$$\frac{\overline{\Delta}_k}{\|\delta\|_2^2} =_d \frac{Z^\top B Z}{\|Z\|_2^2}.$$

Clearly,  $\|Z\|_2^2/p \rightarrow 1$  almost surely, and so it suffices to show that for any  $c < 1$ ,

$$\mathbb{P}\left(\frac{Z^\top B Z}{p} \geq \frac{c k}{\text{tr}(\Sigma)}\right) \rightarrow 1.$$

To see this, first note that if the condition  $\|B\|_{\text{op}}/\text{tr}(B) = o(1)$  holds, then Lemma 1 guarantees

$$\mathbb{P}\left(\frac{Z^\top B Z}{p} \geq \frac{c \text{tr}(B)}{p}\right) \rightarrow 1.$$

Since Lemma 3 ensures that  $\frac{\text{tr}(B)}{p} \geq \frac{k}{\text{tr}(\Sigma)}$ , it remains to verify  $\|B\|_{\text{op}}/\text{tr}(B) = o(1)$ . By Lemma 2,  $\|B\|_{\text{op}} \leq \frac{1}{\lambda_{\min}(\Sigma)}$ , and again, by Lemma 3,  $\text{tr}(B) \geq k \cdot (\frac{p}{\text{tr}(\Sigma)})$ . Consequently, our assumption  $\frac{1}{\lambda_{\min}(\Sigma)k} \frac{\text{tr}(\Sigma)}{p} = o(1)$  gives

$$\frac{\|B\|_{\text{op}}}{\text{tr}(B)} \leq \frac{1}{\lambda_{\min}(\Sigma) \cdot k} \cdot \frac{\text{tr}(\Sigma)}{p} = o(1).$$

□

*Proof of part (ii).* Let  $r := \text{rank}(\Gamma)$ . Since  $\Gamma \succeq 0$  we may write  $\Gamma = V_r V_r^\top$ , where  $V_r \in \mathbb{R}^{p \times r}$  and  $\text{rank}(V_r) = r$ . Let  $B_r := P_k^\top V_r \in \mathbb{R}^{k \times k}$ . By the Woodbury formula,

$$\begin{aligned} (P_k^\top \Sigma P_k)^{-1} &= (P_k^\top (I_{p \times p} + \Gamma) P_k)^{-1} \\ &= (I_{k \times k} + P_k^\top V_r V_r^\top P_k)^{-1} \\ &= I_{k \times k} - \underbrace{B_r (I_{k \times k} + B_r^\top B_r)^{-1} B_r^\top}_{=: C_r}. \end{aligned} \tag{34}$$

Hence,

$$\Delta_k = \delta^\top \underbrace{P_k (I_{k \times k} - C_r) P_k^\top}_{=: G} \delta.$$

Based on the preceding calculations, it can be verified that for almost every  $P_k$ , the matrix  $G$  has operator norm 1, and that at least  $k - r$  eigenvalues of  $G$  are equal to 1. Consequently,

$$\frac{\|\mathbb{E}_{P_k}[G]\|_{\text{op}}}{\text{tr}(\mathbb{E}_{P_k}[G])} \leq \frac{1}{k - r} = o(1),$$

and then using the representation  $\delta/\|\delta\|_2 =_d Z/\|Z\|_2$ , Lemma 1 gives

$$\frac{\overline{\Delta}_k}{\|\delta\|_2^2} \Big/ \frac{\text{tr}(\mathbb{E}_{P_k}[G])}{p} \rightarrow 1 \text{ in probability under } \mathbb{P}_\delta.$$

To simplify  $\text{tr}(\mathbb{E}_{P_k}[G])$ , observe that for almost all  $P_k$ , the non-zero eigenvalues of  $C_r$  are at most 1, since they are of the form  $\lambda/(1 + \lambda)$  where  $\lambda$  is a non-zero eigenvalue of the positive semi-definite matrix  $B_r^\top B_r$ . Furthermore, because  $C_r$  has rank  $r$ , we have  $\text{tr}(\mathbb{E}_{P_k}[C_r]) = o(k)$ , and so

$$\text{tr}(\mathbb{E}_{P_k}[G]) = \mathbb{E}_{P_k}[\text{tr}(I_{k \times k} - C_r)] = k - \mathbb{E}_{P_k}[\text{tr}(C_r)] = k \cdot (1 - o(1)),$$

which implies that

$$\frac{\overline{\Delta}_k}{\|\delta\|_2^2} \Big/ \frac{k}{p} \rightarrow 1 \text{ in probability under } \mathbb{P}_\delta.$$

□

## A.2 Proof of Proposition 2.

It is sufficient to show that the limits  $\overline{\Delta}_k/k \rightarrow 1$  and  $\|\delta\|_2^2/\text{tr}(\Sigma) \rightarrow 1$  hold in probability under  $\mathbb{P}_\delta$ . Since  $\overline{\Delta}_k/\|\delta\|_2$  is invariant under scaling of  $\delta$ , we may write  $\delta = \Sigma^{1/2}Z$  with  $Z \sim N(0, I_{p \times p})$ , and then Lemma 1 immediately gives the second limit. To prove the first limit, we will show that  $\overline{\Delta}_k/k$  converges to 1 in  $L^2(\mathbb{P}_\delta)$ . Expanding out the definition of  $\overline{\Delta}_k$ , we find

$$\overline{\Delta}_k = \mathbb{E}_{P_k} [Z^\top \Sigma^{1/2} P_k (P_k^\top \Sigma P_k)^{-1} P_k^\top \Sigma^{1/2} Z].$$

Notice that for any fixed realization of  $P_k$ , the matrix  $M := \Sigma^{1/2} P_k (P_k^\top \Sigma P_k)^{-1} P_k^\top \Sigma^{1/2}$ , is idempotent with rank  $k$ . Hence, conditional on  $P_k$ , the variable  $Z^\top M Z$  has a  $\chi_k^2$  distribution, which does not depend on  $P_k$ . By integrating over  $\delta$  before  $P_k$ , it follows that  $\mathbb{E}_\delta(\overline{\Delta}_k/k) = 1$ , and consequently, it is enough to show that  $\text{var}_\delta(\overline{\Delta}_k/k) = o(1)$ . By the law of total variance, and using the fact that  $\text{tr}(M) = k$  and  $\|M\|_F^2 = k$  for each realization of  $P_k$ , we have

$$\begin{aligned} \text{var}_\delta(\overline{\Delta}_k/k) &= \frac{1}{k^2} \mathbb{E}_M [\text{var}_Z [Z^\top M Z | M]] + \frac{1}{k^2} \text{var}_M [\mathbb{E}_Z [Z^\top M Z | M]] \\ &= \frac{1}{k^2} \mathbb{E}_M [2 \|M\|_F^2] + \frac{1}{k^2} \text{var}_M [\text{tr}(M)] \\ &= \frac{2k}{k^2} + 0, \end{aligned}$$

which is an  $o(1)$  quantity as required.

## B Proof of Theorem 1

Let  $\tau := \frac{n_1+n_2}{n_1 n_2}$  and  $Z \sim N(0, I_{p \times p})$ . Under the null hypothesis that  $\delta = 0$ , we have  $\bar{X} - \bar{Y} \stackrel{d}{=} \sqrt{\tau} \Sigma^{1/2} Z$ , and consequently,

$$T_k^2 \stackrel{d}{=} Z^\top \underbrace{\Sigma^{1/2} \mathbb{E}_{P_k} [P_k (P_k^\top \widehat{\Sigma} P_k)^{-1} P_k] \Sigma^{1/2}}_{=: A} Z,$$

which gives the convenient representation  $\bar{T}_k^2 = Z^\top A Z$ . Note that  $A$  is a random matrix, and that we may take  $\bar{X} - \bar{Y}$  and  $\widehat{\Sigma}$  to be independent for Gaussian data [28, p.77]. Consequently, we may assume that  $Z$  and  $A$  are independent. Our overall strategy is to work conditionally on  $A$ , and use the representation

$$\mathbb{P} \left( \frac{Z^\top A Z - \bar{\mu}_n}{\bar{\sigma}_n} \leq x \right) = \mathbb{E}_A \mathbb{P}_Z \left( \frac{Z^\top A Z - \bar{\mu}_n}{\bar{\sigma}_n} \leq x \mid A \right),$$

where  $x \in \mathbb{R}$ . To demonstrate the asymptotic normality of  $Z^\top AZ$ , we will show in Section B.4 that the limit  $\|A\|_{\text{op}} / \|A\|_F = o_{\mathbb{P}_A}(1)$  holds. This implies the Lyupanov condition [29], which in turn implies the Lindeberg condition, and it then follows [30, Sec. 3.2] that<sup>4</sup>

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}_Z \left( \frac{Z^\top AZ - \text{tr}(A)}{\sqrt{2}\|A\|_F} \leq x \mid A \right) - \Phi(x) \right| = o_{\mathbb{P}_A}(1). \quad (35)$$

The next step is to show that  $\text{tr}(A)$  and  $\|A\|_F$  can be replaced with deterministic counterparts  $\bar{\mu}_n := \frac{y_n}{1-y_n}n$  and  $\bar{\sigma}_n := \sqrt{\frac{2y_n}{(1-y_n)^3}}\sqrt{n}$ . More precisely, we show in Sections B.2 and B.3 (respectively) that

$$\text{tr}(A) - \bar{\mu}_n = o_{\mathbb{P}_A}(\sqrt{n}), \quad \text{and} \quad (36a)$$

$$\|A\|_F - \bar{\sigma}_n = o_{\mathbb{P}_A}(\sqrt{n}). \quad (36b)$$

Substituting the previous formulas into the limit (35), it follows that

$$\mathbb{P} \left( \frac{Z^\top AZ - \bar{\mu}_n}{\bar{\sigma}_n} \leq x \mid A \right) - \Phi(x) = o_{\mathbb{P}_A},$$

and the central limit theorem (4) follows from the dominated convergence theorem.  $\square$

## B.1 Matrix Reduction of $A$ .

Before dealing directly with  $\text{tr}(A)$  and  $\|A\|$  in Sections B.2, B.3 and B.4, we will need a lemma that reduces the matrix  $A := \Sigma^{1/2} \mathbb{E}_{P_k} [P_k (P_k^\top \hat{\Sigma} P_k)^{-1} P_k^\top] \Sigma^{1/2}$  to a simpler form. In particular, let us recall the definition of the *thin QR factorization* [17, p.230]: namely, if  $M \in \mathbb{R}^{p \times k}$  is full rank with  $k \leq p$ , then there exist unique matrices  $Q \in \mathbb{R}^{p \times k}$  and  $R \in \mathbb{R}^{k \times k}$  such that  $M = QR$ , where  $Q$  has orthonormal columns, and  $R$  is an invertible upper triangular matrix with positive diagonal entries.

**Lemma 4.** *Let  $P_k \in \mathbb{R}^{p \times k}$  be a random matrix that is full rank with probability 1, and let  $\Sigma^{1/2} P_k = QR$  be a thin QR factorization. Then we have*

$$A \stackrel{d}{=} n \mathbb{E}_Q [Q(Q^\top W_p Q)^{-1} Q^\top], \quad (37)$$

where  $W_p \sim W_p(n, I_{p \times p})$  is a white Wishart matrix that is independent of  $Q$ .

Since  $\hat{\Sigma} \stackrel{d}{=} \Sigma^{1/2} W_p \Sigma^{1/2}$ , this claim (37) follows after substituting the formula  $P_k = \Sigma^{-1/2} QR$  into the definition of  $A$  and making use of several cancellations.

## B.2 Asymptotic formula for $\text{tr}(A)$

In this section, we verify the limit (36a). Define the sequence of random variables  $X_n := \frac{1}{\sqrt{n}}(\text{tr}(A) - \bar{\mu}_n)$ . It suffices to show that  $\mathbb{E}|X_n| \rightarrow 0$ . Using the cyclic property of trace, in conjunction with equation (37) and  $Q^\top Q = I_{k \times k}$ , we obtain  $\text{tr}(A) = n \mathbb{E}_Q [\text{tr}((Q^\top W_p Q)^{-1})]$ . Using this relation, we have

$$\mathbb{E}|X_n| = \sqrt{n} \mathbb{E}_{W_p} \left| \mathbb{E}_Q [\text{tr}((Q^\top W_p Q)^{-1})] - \frac{y_n}{1-y_n} \right|.$$

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<sup>4</sup>The uniformity of the limit (35) will be useful later on in Section 3.1.



By Jensen's inequality, we obtain an upper bound by extracting  $\mathbb{E}_Q$  from the absolute value, and then we exchange  $\mathbb{E}_{W_p}$  and  $\mathbb{E}_Q$  to obtain

$$\mathbb{E}|X_n| \leq \sqrt{n} \mathbb{E}_Q \mathbb{E}_{W_p} \left| \text{tr}((Q^\top W_p Q)^{-1}) - \frac{y_n}{1-y_n} \right|.$$

Since  $Q$  and  $W_p$  are independent, we may fix  $Q$  without affecting the distribution of  $W_p$ , and we also note that for a fixed  $Q$ , the matrix  $Q^\top W_p Q$  is distributed as a white Wishart matrix  $W_k$  whose distribution does not depend on  $Q$ . Consequently, if we take the expectation over  $W_p$  first, then the expectation over  $Q$  becomes irrelevant, and we have

$$\mathbb{E}|X_n| \leq \sqrt{n} \mathbb{E}_{W_k} \left| \text{tr}(W_k^{-1}) - \frac{y_n}{1-y_n} \right|.$$

By Lemma 2.2 in the paper [5], the variable  $Y_n := \sqrt{n} \left( \text{tr}(W_k^{-1}) - \frac{y_n}{1-y_n} \right)$  tends to 0 in probability, and so it remains to check that the sequence  $Y_n$  is uniformly integrable to ensure that  $\mathbb{E}|Y_n| = o(1)$ , which will complete the proof. By Theorem 2.4.14 in [31, p.257], mean and variance of  $Y_n$  are both  $\mathcal{O}(1)$ , which means that the sequence  $Y_n$  is bounded in  $L^2$ , and hence uniformly integrable.

### B.3 Asymptotic formula for $\|A\|_F$

In this section, we verify the limit (36b). Define the sequence of variables  $X_n := \frac{1}{\sqrt{n}} (\|A\|_F - \bar{\sigma}_n)$ . It suffices to show that  $\mathbb{E}|X_n| \rightarrow 0$ . First observe that by the reduced formula (37) for  $A$ , we have

$$\mathbb{E}|X_n| = \mathbb{E}_{W_p} \left| \sqrt{n} \left\| \mathbb{E}_Q \left[ Q(Q^\top W_p Q)^{-1} Q^\top \right] \right\|_F - \sqrt{\frac{y_n}{(1-y_n)^3}} \right|.$$

Applying Jensen's inequality twice, and exchanging the order of  $\mathbb{E}_Q$  and  $\mathbb{E}_{W_p}$  yields

$$\mathbb{E}|X_n| \leq \mathbb{E}_Q \mathbb{E}_{W_p} \left| \sqrt{n} \left\| Q(Q^\top W_p Q)^{-1} Q^\top \right\|_F - \sqrt{\frac{y_n}{(1-y_n)^3}} \right|.$$

Since the non-zero eigenvalues of  $Q(Q^\top W_p Q)^{-1} Q^\top$  are the same as those of  $(Q^\top W_p Q)^{-1}$ , these matrices have the same Frobenius norm, and so

$$\mathbb{E}|X_n| \leq \mathbb{E}_Q \mathbb{E}_{W_p} \left| \sqrt{n} \left\| (Q^\top W_p Q)^{-1} \right\|_F - \sqrt{\frac{y_n}{(1-y_n)^3}} \right|.$$

Applying the same reasoning as in the previous subsection, we may replace  $(Q^\top W_p Q)^{-1}$  with  $W_k \sim W_k(n, I_{k \times k})$  when working under  $\mathbb{E}_{W_p}$ , and then the expectation over  $Q$  becomes irrelevant. Putting together the pieces, we find

$$\mathbb{E}|X_n| \leq \mathbb{E}_{W_k} \left| \sqrt{n} \left\| W_k^{-1} \right\|_F - \sqrt{\frac{y_n}{(1-y_n)^3}} \right|.$$

By Lemma 2.2 in the paper [5], the random variable  $Y_n := \sqrt{n} \left\| W_k^{-1} \right\|_F - \sqrt{\frac{a}{(1-a)^3}}$  tends to 0 in probability, and so it remains to check that the sequence  $Y_n$  is uniformly integrable to ensure that  $\mathbb{E}|Y_n| = o(1)$ , which will complete the proof. By Theorem 2.4.14 in [31], it follows that  $n \mathbb{E}_{W_k} \left\| W_k^{-1} \right\|_F^2 = \mathcal{O}(1)$ , which implies that the sequence  $Y_n$  is bounded in  $L^2$ , and hence uniformly integrable.

#### B.4 Verification of $\|A\|_{\text{op}} / \|A\|_F = o_{\mathbb{P}_A}(1)$ .

The asymptotic formula for  $\|A\|_F$  proven in Section B.3 shows that  $\frac{1}{\sqrt{n}} \|A\|_F$  tends to a positive constant in probability under  $\mathbb{P}_A$ . Consequently, to verify  $\|A\|_{\text{op}} / \|A\|_F = o_{\mathbb{P}_A}(1)$ , it is enough to show that  $\frac{1}{\sqrt{n}} \|A\|_{\text{op}}$  tends to 0 in probability, and this is implied by the following lemma.

**Lemma 5.** *The matrix  $A$  satisfies  $\frac{1}{\sqrt{n}} \mathbb{E} \|A\|_{\text{op}} \rightarrow 0$ .*

*Proof.* By Jensen's inequality and the fact that the matrices  $Q(Q^\top W_p Q)^{-1} Q^\top$  and  $(Q^\top W_p Q)^{-1}$  have the same non-zero eigenvalues,

$$\begin{aligned} \frac{1}{n} \mathbb{E} \|A\|_{\text{op}} &= \mathbb{E}_{W_p} \left\| \mathbb{E}_Q [Q(Q^\top W_p Q)^{-1} Q^\top] \right\|_{\text{op}} \\ &\leq \mathbb{E}_{W_p} \mathbb{E}_Q \left\| Q(Q^\top W_p Q)^{-1} Q^\top \right\|_{\text{op}} \\ &= \mathbb{E}_Q \mathbb{E}_{W_p} \left\| (Q^\top W_p Q)^{-1} \right\|_{\text{op}} \\ &= \mathbb{E}_{W_k} \left\| W_k^{-1} \right\|_{\text{op}}. \end{aligned} \tag{38}$$

Adjusting by the factor of  $1/\sqrt{n}$ , we have

$$\frac{1}{\sqrt{n}} \mathbb{E} \|A\|_{\text{op}} \leq \frac{1}{\sqrt{n}} \mathbb{E}_{W_k} \left\| n W_k^{-1} \right\|_{\text{op}}.$$

For any  $\epsilon \in (0, c_*)$ , define the constant  $c_* := (1 - \sqrt{y})^2$ , and the event  $E := \{\lambda_{\min}(\frac{1}{n} W_k) \geq c_* - \epsilon\}$ . We now split the expectation  $\mathbb{E}_{W_k} \left\| n W_k^{-1} \right\|_{\text{op}}$  along  $E$  and  $E^c$ ,

$$\frac{1}{\sqrt{n}} \mathbb{E}_{W_k} \left\| n W_k^{-1} \right\|_{\text{op}} = \frac{1}{\sqrt{n}} \mathbb{E}_{W_k} \left[ \left\| n W_k^{-1} \right\|_{\text{op}} \cdot 1_E \right] + \frac{1}{\sqrt{n}} \mathbb{E}_{W_k} \left[ \left\| n W_k^{-1} \right\|_{\text{op}} \cdot 1_{E^c} \right].$$

On the set  $E$ , the random variable  $\left\| n W_k^{-1} \right\|_{\text{op}}$  is bounded by the constant  $1/(c_* - \epsilon)$ , and so the first piece tends to 0 due to the factor of  $\frac{1}{\sqrt{n}}$ . Next, we upper bound the second piece in several steps. First we replace the operator norm with the Frobenius norm, and then use the Cauchy-Schwarz inequality in conjunction with the formula  $\mathbb{E}_{W_k} \left\| W_k^{-1} \right\|_F^2 = \mathcal{O}(1/n)$  from Theorem 2.4.14 of [31, p. 256].

$$\begin{aligned} \frac{1}{\sqrt{n}} \mathbb{E}_{W_k} \left[ \left\| n W_k^{-1} \right\|_{\text{op}} 1_{E^c} \right] &\leq \frac{1}{\sqrt{n}} \mathbb{E}_{W_k} \left[ \left\| n W_k^{-1} \right\|_F 1_{E^c} \right] \\ &\leq \frac{1}{\sqrt{n}} \sqrt{\mathbb{E}_{W_k} \left\| n W_k^{-1} \right\|_F^2} \cdot \sqrt{\mathbb{P}(E^c)} \\ &= \frac{1}{\sqrt{n}} \mathcal{O}(\sqrt{n}) \cdot \sqrt{\mathbb{P}(E^c)} \\ &\lesssim \sqrt{\mathbb{P}(E^c)}. \end{aligned} \tag{39}$$

Since  $\lambda_{\min}(\frac{1}{n} W_k) \rightarrow c_*$  almost surely [32], we have  $\mathbb{P}(E^c) \rightarrow 0$ , which proves the claim.  $\square$

## C Proof of Theorem 2

Working under the alternative hypothesis, we seek the limiting value of the power function  $\beta(\theta) = \mathbb{P}(\frac{1}{n} \bar{T}_k^2 \geq t_\alpha)$ . Since  $\delta \neq 0$ , the test statistic splits into three terms. Define  $\tau := \frac{n_1+n_2}{n_1 n_2}$ , and recall

$$\bar{T}_k^2 = (1/\tau) \cdot (\bar{X} - \bar{Y})^\top \mathbb{E}_{P_k} [P_k (P_k^\top M P)^{-1} P_k^\top] (\bar{X} - \bar{Y}),$$

where

$$\bar{X} - \bar{Y} \stackrel{d}{=} \sqrt{\tau} \Sigma^{1/2} Z + \delta,$$

and  $Z$  is a standard Gaussian  $p$ -vector. Expanding the definition of  $\bar{T}_k^2$ , and adjusting by a factor of  $\frac{1}{n}$ , we have the decomposition  $\frac{1}{n} \bar{T}_k^2 = \text{I} + \text{II} + \text{III}$ , where

$$\text{I} := \frac{1}{n} Z^\top A Z \tag{40a}$$

$$\text{II} := 2 \frac{1}{n \sqrt{\tau}} Z^\top A \Sigma^{-1/2} \delta \tag{40b}$$

$$\text{III} := \frac{1}{n \tau} \delta^\top \Sigma^{-1/2} A \Sigma^{-1/2} \delta. \tag{40c}$$

Here the reader should recall the definition of the matrix  $A$  from Appendix B—namely

$$A := \Sigma^{1/2} \mathbb{E}_{P_k} [P_k (P_k^\top \widehat{\Sigma} P)^{-1} P_k^\top] \Sigma^{1/2}.$$

With these definitions in hand, we will work conditionally on  $A$  and consider

$$\mathbb{P}(T_k^2 \geq t_\alpha) = \mathbb{E}_A \mathbb{P}_Z(\text{I} \geq \frac{1}{n} t_\alpha - \text{II} - \text{III} \mid A).$$

Studentizing with  $\text{tr}(\frac{1}{n} A)$  and  $\|\frac{1}{n} A\|_F$ , and multiplying top and bottom by  $\sqrt{n}$ ,

$$\mathbb{P}(T_k^2 \geq t_\alpha) = \mathbb{E}_A \mathbb{P}_Z\left(\frac{Z^\top (\frac{1}{n} A) Z - \text{tr}(\frac{1}{n} A)}{\sqrt{2} \|\frac{1}{n} A\|_F} \geq \frac{\sqrt{n}(\frac{1}{n} t_\alpha - \text{tr}(\frac{1}{n} A) - \text{II} - \text{III})}{\sqrt{n} \sqrt{2} \|\frac{1}{n} A\|_F} \mid A\right). \tag{41}$$

Recall the definition of the critical value  $t_\alpha := \frac{y_n}{1-y_n} n + \sqrt{\frac{2y_n}{(1-y_n)^3}} \sqrt{n} z_{1-\alpha}$ , and define the numerical sequence  $\ell_n := \frac{1}{\tau n} \left( \frac{n}{n-k-1} \right) \bar{\Delta}_k$ . In Sections C.1 and C.2, we establish the limits

$$\mathbb{P}_Z(\sqrt{n} |\text{II}| > \epsilon \mid A) = o_{\mathbb{P}_A}(1), \quad \text{and} \quad \sqrt{n} (\text{III} - \ell_n) = o_{\mathbb{P}_A}(1),$$

where  $\epsilon > 0$ . Substituting the limits (36a) and (36b) into equation (41) yields

$$\mathbb{P}(T_k^2 \geq t_\alpha) = \mathbb{E}_A \mathbb{P}_Z\left(\frac{Z^\top A Z - \text{tr}(A)}{\sqrt{2} \|A\|_F} \geq \frac{\sqrt{n} \left( \sqrt{\frac{2y_n}{(1-y_n)^3}} z_{1-\alpha} - \ell_n - \text{II} \right)}{\sqrt{\frac{2y_n}{(1-y_n)^3}}} + o_{\mathbb{P}_A}(1) \mid A\right). \tag{42}$$

By the limit (35), it follows that

$$\mathbb{P}_Z\left(\frac{Z^\top A Z - \text{tr}(A)}{\sqrt{2} \|A\|_F} \geq z_{1-\alpha} - \sqrt{n} \sqrt{\frac{(1-y_n)^3}{2y_n}} (\ell_n + \text{II}) + o_{\mathbb{P}_A}(1) \mid A\right) = \Phi\left(-z_{1-\alpha} + \sqrt{n} \sqrt{\frac{(1-y_n)^3}{2y_n}} \ell_n\right) + o_{\mathbb{P}_A}(1),$$

where the error term  $o_{\mathbb{P}_A}(1)$  on the right hand side is bounded by 1. Integrating over  $A$  and applying the dominated convergence theorem,

$$\mathbb{P}\left(T_k^2 \geq t_\alpha\right) = \Phi\left(-z_{1-\alpha} + \sqrt{n}\sqrt{\frac{(1-y_n)^3}{2y_n}}\ell_n\right) + o(1).$$

Using the assumptions  $y_n = \frac{k}{n} = a + o(\frac{1}{\sqrt{n}})$ , and  $\frac{n_1}{n} = b + o(\frac{1}{\sqrt{n}})$ , we conclude

$$\mathbb{P}\left(T_k^2 \geq t_\alpha\right) = \Phi\left(-z_{1-\alpha} + b(1-b) \cdot \sqrt{\frac{1-y}{2y}} \cdot \bar{\Delta}_k \sqrt{n}\right) + o(1).$$

□

### C.1 The cross term II tends to 0.

In this section, we prove that under the conditions Theorem 2, for any  $\epsilon > 0$ , we have the limit

$$\mathbb{P}_Z\left(|\sqrt{n}\text{II}| > \epsilon \mid A\right) = o_{\mathbb{P}_A}(1). \quad (43)$$

It suffices to show that  $\mathbb{E}_A \mathbb{P}_Z\left(|\sqrt{n}\text{II}| > \epsilon \mid A\right) = \mathbb{P}\left(|\sqrt{n}\text{II}| > \epsilon\right) = o(1)$ . Since  $\sqrt{n}\text{II}$  has mean 0, i.e.  $\mathbb{E}_A \mathbb{E}_Z[\sqrt{n}\text{II}] = \mathbb{E}_A[0] = 0$ , Chebyshev's inequality implies that the last condition will hold if  $\text{var}[\sqrt{n}\text{II}] = o(1)$ . By the law of total variance, we have

$$\text{var}[\sqrt{n}\text{II}] = \mathbb{E}_A \text{var}_Z[\sqrt{n}\text{II} \mid A] + \text{var}_A[\mathbb{E}_Z[\sqrt{n}\text{II} \mid A]],$$

where we note the the second term is 0. Writing out the first term, we have

$$\begin{aligned} \text{var}[\sqrt{n}\text{II}] &= \mathbb{E}_A \text{var}_Z[\sqrt{n}\text{II} \mid A] \\ &= \mathbb{E}_A\left[\frac{4}{n\tau} \delta^\top \Sigma^{-1/2} A^2 \Sigma^{-1/2} \delta\right] \\ &= \mathbb{E}_{W_p}\left[\frac{4}{n\tau} \left\|\mathbb{E}_Q\left[Q(Q^\top \frac{1}{n} W_p Q)^{-1} Q^\top\right] \Sigma^{-1/2} \delta\right\|_2^2\right] \\ &\leq \frac{4}{n\tau} \mathbb{E}_Q \mathbb{E}_{W_p}\left[\left\|Q(Q^\top \frac{1}{n} W_p Q)^{-1} Q^\top \Sigma^{-1/2} \delta\right\|_2^2\right], \end{aligned}$$

where the last step follows by Jensen's inequality. We then use iterated expectation to write

$$\begin{aligned} \text{var}[\sqrt{n}\text{II}] &= \frac{4}{n\tau} \mathbb{E}_Q\left[\delta^\top \Sigma^{-1/2} Q \mathbb{E}_{W_p}\left[(Q^\top \frac{1}{n} W_p Q)^{-1}\right] Q^\top \Sigma^{-1/2} \delta\right] \\ &= \frac{4}{n\tau} \mathbb{E}_Q\left[\delta^\top \Sigma^{-1/2} Q \left(c_n I_{k \times k}\right) Q^\top \Sigma^{-1/2} \delta\right] \end{aligned}$$

where the last step uses the fact that  $\mathbb{E}_{W_k}[(\frac{1}{n} W_k)^{-1}] = c_n I_{k \times k}$  with  $c_n = \frac{n}{n-k-1}$ . Since the maximum singular value of  $Q$  is one, we have

$$\begin{aligned} \text{var}[\sqrt{n}\text{II}] &= \frac{4c_n}{n\tau} \mathbb{E}_Q\left[\left\|Q^\top \Sigma^{-1/2} \delta\right\|_2^2\right] \leq \frac{4c_n}{n\tau} \|\Sigma^{-1/2} \delta\|_2^2 \\ &= \mathcal{O}(\delta^\top \Sigma^{-1} \delta), \end{aligned}$$

where the final step uses the facts that  $c_n = \mathcal{O}(1)$  and  $\frac{1}{n\tau} = \mathcal{O}(1)$ . This completes the proof, since  $\delta^T \Sigma^{-1} \delta = o(1)$  by our local alternative assumption.

## C.2 Limiting value of III.

Before we study the limiting value of III, we record an elementary lemma.

**Lemma 6.** *Let  $W_k \sim W_k(n, I_{k \times k})$  be a white  $k \times k$  Wishart matrix, and let  $\chi_{n-k+1}^2$  be a chi-square variable on  $n - k + 1$  degrees of freedom. If  $u \in \mathbb{R}^k$  is a fixed unit vector, and  $n > k + 3$ , then*

$$u^\top W_k^{-1} u \stackrel{d}{=} \frac{1}{\chi_{n-k+1}^2},$$

which has mean  $\frac{1}{n-k-1}$  and variance  $\frac{2}{(n-k-1)^2(n-k-3)}$ .

*Proof.* By Corollary 2.4.13.1 in [31, p. 256], the random variable  $(u^\top W_k^{-1} u)^{-1}$  is distributed according to  $W_1(n - k + 1, (u^\top I_{k \times k} u)^{-1})$ , which is the same as the chi-square distribution on  $n - k + 1$  degrees of freedom.  $\square$

In order to give the limiting value of III, we now define some notation. Let  $v := \Sigma^{-1/2} \delta$ , and recall the definition  $\ell_n := \frac{1}{\tau n} \left( \frac{n}{n-k-1} \right) \overline{\Delta}_k$ . Tracing back through the definition of  $Q$  in Lemma 4, it is straightforward to verify that  $\mathbb{E}_Q \|Q^\top v\|_2^2 = \overline{\Delta}_k$ , and so we may also write

$$\ell_n = \frac{1}{\tau n} \left( \frac{n}{n-k-1} \right) \mathbb{E}_Q \|Q^\top v\|_2^2,$$

which will turn out to be the asymptotic equivalent of III. We now show that under the conditions of Theorem 2

$$\sqrt{n}(\text{III} - \ell_n) = o_{\mathbb{P}_A}(1).$$

Recalling the definition  $\text{III} := \frac{1}{n\tau} \delta^\top \Sigma^{-1/2} A \Sigma^{-1/2} \delta$ , we define the sequence of random variables  $V_n := \sqrt{n}(\text{III} - \ell_n)$ . Since

$$\mathbb{E}[A] = n \mathbb{E}_Q [Q \mathbb{E}_{W_p} [(Q^\top W_p Q)^{-1}] Q^\top] = \mathbb{E}_Q [Q \frac{n}{n-k-1} I_{k \times k} Q^\top] = \frac{n}{n-k-1} \mathbb{E}_Q [Q Q^\top], \quad (44)$$

we see that  $\mathbb{E}[V_n] = 0$ . Consequently, it suffices to show that  $\text{var}[V_n] = \mathbb{E}[V_n^2] \rightarrow 0$ . Note that

$$V_n^2 = n \left( \frac{1}{n\tau} v^\top E_Q \left[ n Q (Q^\top W_p Q)^{-1} Q^\top \right] v - \ell_n \right)^2.$$

To obtain an upper bound on the second moment, we apply Jensen's inequality, and switch the order of integration, thereby obtaining

$$\begin{aligned} \mathbb{E}[V_n^2] &= n \mathbb{E}_{W_p} \left[ \left( \frac{1}{n\tau} \mathbb{E}_Q \left[ n v^\top Q (Q^\top W_p Q)^{-1} Q^\top v - \frac{1}{\tau n} \left( \frac{n}{n-k-1} \right) \|Q^\top v\|_2^2 \right] \right)^2 \right], \\ &\leq n^3 \mathbb{E}_Q \mathbb{E}_{W_p} \left[ \left( \frac{1}{n\tau} v^\top \left[ n Q (Q^\top W_p Q)^{-1} Q^\top \right] v - \frac{1}{\tau n} \left( \frac{n}{n-k-1} \right) \|Q^\top v\|_2^2 \right)^2 \right]. \end{aligned}$$

Since  $W_p$  and  $Q$  are independent, we may assume that  $Q$  is fixed when working under  $\mathbb{E}_{W_p}$ . Define the unit vector  $u = \frac{Q^\top v}{\|Q^\top v\|_2}$ , and recall that for a fixed  $Q$ , the matrix  $Q^\top W_p Q$  is distributed as  $W_k \sim W_k(n, I_{k \times k})$ . Moving  $\|Q^\top v\|_2^2$  between  $\mathbb{E}_Q$  and  $\mathbb{E}_{W_k}$ , and moving  $\frac{n}{\tau n}$  outside both expectations, we have

$$\mathbb{E}[V_n^2] \leq \frac{n^3}{(n\tau)^2} \mathbb{E}_Q \left[ \|Q^\top v\|_2^4 \mathbb{E}_{W_k} \left[ \left( u^\top W_k^{-1} u - \frac{1}{n-k-1} \right)^2 \right] \right]$$

Since the expectation under  $\mathbb{E}_{W_k}$  does not depend on  $Q$ , we may factor the right hand side into a product of two expectations. After moving the factor of  $n^3$  in front of  $\mathbb{E}_{W_k}$ , this gives

$$\mathbb{E}[V_n^2] \leq \frac{1}{(n\tau)^2} \mathbb{E}_Q [\|Q^\top v\|_2^4] \cdot n^3 \cdot \mathbb{E}_{W_k} \left[ \left( u^\top W_k^{-1} u - \frac{1}{n-k-1} \right)^2 \right]$$

By the lemma,  $u^\top W_k^{-1} u \sim \frac{1}{\chi_{n-k+1}^2}$ , which has mean  $\frac{1}{n-k-1}$  and variance  $\frac{2}{(n-k-1)^2(n-k-3)}$ . Consequently

$$n^3 \mathbb{E}_{W_k} \left[ \left( u^\top W_k^{-1} u - \frac{1}{n-k-1} \right)^2 \right] = \mathcal{O}(1).$$

We also recall that  $\frac{1}{(n\tau)^2} \rightarrow b^2(1-b)^2$ , and that under our local alternative,

$$\mathbb{E}_Q [\|Q^\top v\|_2^4] \leq \|v\|_2^4 = (\delta^\top \Sigma^{-1} \delta)^2 = o(1),$$

which completes the proof that  $\mathbb{E}[V_n^2] \rightarrow 0$ .

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